Stochastic differential equations driven by fractal processes

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Zusammenfassung

Die vorliegende Arbeit befasst sich mit der Studie von gewöhnlichen stochastischen Differentialgleichungen (SDE's) mit fraktalem Rauschen. Wir beweisen Existenz- und Eindeutigkeitsaussagen für Lösungen solcher Gleichungen. Dabei werden diese pfadweise betrachtet. Im Gegensatz zur klassischen stochastischen Analysis sind die dabei auftretenden Prozesse keine Semimartingale und es stehen in dieser Situation daher keine Integrale vom Itô-typ zur Verfügung. Die Integrale sind als verallgemeinerte stochastische Vorwärtsintegrale zu verstehen. Wir betrachten die SDE's im $\mathbb{R}^n$ mit zeitabhängigen (nicht notwendigerweise adaptierten) zufälligen und nichtlinearen Koeffizienten. Dabei besitzt einer der treibenden Prozesse Z^0 eine endliche verallgemeinerte quadratische Variation und die anderen Prozesse $Z^1, ..., Z^m$ haben Pfade in gebrochen Sobolevräumen der Ordnung größer als 1/2. Als Beispiel hierfür kann man eine Brownsche Bewegung und m gebrochene Brownsche Bewegungen ansehen. Mithilfe eines Doss-Sussman Ansatzes, eines gleichmäßigen lokalen Kontraktionsprinzips und unter geeigneten Bedingungen an die Koeffizienten erhalten wir die gewünschten Aussagen bezüglich der Existenz und Eindeutigkeit. Desweiteren ergeben sich optimale Eigenschaften bezüglich der Hölderregularität für solche Lösungen, sofern die Pfade der treibenden Prozesse selber gewisse Höldereigenschaften besitzen, sowohl für den Fall $Z^0 \equiv 0$ als auch $Z^0 \not\equiv 0$. Der Hölderexponent der Lösung entspricht jeweils dem niedrigsten Hölderexponenten der Rauschterme.

Abschließend betrachten wir stochastische Differentialgleichungen des obigen Typs aber mit einem zusätzlichen treibenden endlichen Sprungprozess. Als Beispiel hierfür kann ein zusammengesetzter Poisson Prozess dienen. Auch

für diesen Fall erhalten wir Aussagen über die Existenz und Eindeutigkeit bezüglich der Lösungen.

Abstract

The present thesis deals with the study of ordinary stochastic differential equations (SDE's) with fractal noises. We prove existence and uniqueness results for solutions of such equations, considered in a pathwise sense. In contrast to classical stochastic analysis the involved processes are not semimartingales and so there are in this situation no integrals of Itô-type available. The integrals are to be understood as generalized stochastic forward integrals. We consider SDE's in $\mathbb{R}^n$ with time dependent (not necessarily adapted) random and nonlinear coefficients. One of the driving processes Z^0 admits a finite generalized quadratic variation and the other processes $Z^1, ..., Z^m$ have sample paths in some fractional Sobolev space of order greater than $1/2$. An example would be one Brownian motion and m fractional Brownian motions. Using a Doss-Sussman-type approach, a uniform local contraction principle and under some suitable conditions on the coefficients, we obtain the desired statements regarding the existence and uniqueness. Furthermore we receive optimal Hölder regularity properties for such solutions, if the paths of the driving processes themselves have certain Hölder properties, for both the cases $Z^0 \equiv 0$ and $Z^0 \not\equiv 0$. The Hölder exponent of the solution is the same as the lowest Hölder exponent of the noises.

Finally, we consider stochastic differential equations of the above type but with an additional driving process, a finite jump process. In particular, a compound Poisson process can be treated. Also in that case we obtain results about existence and uniqueness of the solutions.

Contents

Chapter 1

Introduction

The central topic of this thesis is a pathwise approach to stochastic differential equations driven by certain random noises.

During the last decades stochastic analysis became more and more important. It mainly deals with properties of stochastic integrals. In contrast to classical analysis, integrators and integrands are not deterministic functions, but stochastic processes. Consequently, the value of the integral is a random variable rather than a real number. In the natural sciences stochastic processes are often used to model phenomena, such as in financial mathematics for valuing options, in biology to describe population trends or in physics to explain the random movement of particles.
In the 1940's the Japanese mathematician Kiyosi Itô developed the foundations for stochastic analysis. He gave a rigorous definition for integrals of the type

$$\int_0^t Y(s)\, dS(s)$$

if the stochastic process $S = (S(t))_{t\geq 0}$ is a semimartingale and the process $Y = (Y(t))_{t\geq 0}$ is adapted and left continuous. Independently of Itô the French mathematician Doeblin found some years earlier similar results, which were not published until 2000. Starting with this definition it was possible to introduce stochastic differential equations (SDE's) of the form

$$\begin{aligned} dX(t) &= a(X(t),t)\,dt + b(X(t),t)\,dS(t) \\ X(0) &= X_0 \end{aligned}$$

that are defined via the corresponding integral equation

$$X(t) = X_0 + \int_0^t a(X(s),s)\,ds + \int_0^t b(X(s),s)\,dS(s).$$

Here the first integral is a usual Riemann-Stieltjes integral and the second the so called Itô integral. There is a large theory about conditions on the coefficient functions to obtain existence and uniqueness of a solution for the stochastic differential equations related to semimartingales (see for example the books of Applebaum [1], Øksendal [23] and Protter [24]).
One of the most popular stochastic differential equations of that type is

$$\begin{aligned} dX(t) &= \mu X(t)\,dt + \sigma X(t)\,dW(t) \\ X(0) &= 1, \end{aligned}$$

where the process W is a Brownian motion or sometimes called a Wiener process. The solution to this equation is the geometric Brownian motion

$$X(t) = X_0 \exp\big\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\big\},$$

which is used to model stock prices in the Black-Scholes model.
In recent times other processes then semimartingales came into focus, for example the fractional Brownian motion. This is a centered Gaussian process with covariance function

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), s,t \geq 0,$$

with some Hurst parameter $H \in (0,1)$. If $H = 1/2$ we obtain the Brownian motion. For $H \neq 1/2$ the fractional Brownian motion is no semimartingale and Itô calculus can not applied. This processes can be used to model phenomena such as long and short range dependence in the Black-Scholes model and also other models.

In this thesis we follow the approach introduced by Zähle in [39] and [42], where one of the driving processes Z^0 admits a finite generalized quadratic variation and the other processes $Z^1, ..., Z^m$ have sample paths in some fractional Sobolev space of order greater than 1/2 and the integrals are to be understood as generalized stochastic forward integrals. For $Z^1, ..., Z^m$ they coincide with the generalized Lebesgue-Stieltjes integral. The special case when Z^0 is a Brownian motion and $Z^1, ..., Z^m$ are fractional Brownian motions have been investigated by Guerra and Nualart in [10]. The authors proved pathwise global existence and uniqueness of a solution under the assumption that the processes are independent. Without the independece assumption this result was obtained by Mishura et al. [18] and [19]. In the general case there was proven in [39] and [42] existence and uniqueness of a solution but only locally. The aim of this thesis is to extend this approach to obtain global solutions. Under some suitable conditions on the coefficient functions and using a Doss-Sussman-type approach and a uniform local contraction principle we infer a pathwise global solution to this type of stochastic differential equations.

We proceed as follows. After the introduction we introduce some background material about fractional derivatives $D^\alpha_{a+}, D^\alpha_{b-}$ and fractional integrals $I^\alpha_{a+}, I^\alpha_{b-}$ on some finite interval $[a, b] \subset \mathbb{R}$. Further we recall the definitions of the fractional Sobolev spaces $W^\alpha_2(a, b)$ on $[a, b]$, which are most appropriate to our approach. In [38] the generalized Lebesgue-Stieltjes integral of Section 2.4 was defined in terms of fractional calculus. Namely

$$\int_a^b f(x)\, dg(x) = (-1)^\alpha \int_a^b D^\alpha_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x)\, dx + f(a+)(g(b-) - g(a+))$$

if the functions f_{a+} and g_{b-} are elements of the Liouville spaces $I^\alpha_{a+}(L_2(a, b))$ and $I^{1-\alpha}_{b-}(L_2(a, b))$ for some $0 < \alpha < 1$, where $f_{a+}(x) = \mathbf{1}_{(a,b)}(x)(f(x) - f(a+))$ and $g_{b-}(x) = \mathbf{1}_{(a,b)}(x)(g(x) - g(b-))$. Also we derive some norm estimates for this integral in the fractional sobolev space $W^\beta_{2,\infty}(a, b)$ for $\beta > 1/2$ (Theorem 2.8).

In Chapter 3 we introduce the integral operator

$$Af := x_0 + \int_{t_0}^{(\cdot)} a(f, \varphi)\, dg$$

for a parameter function $\varphi \in W_{2,\infty}^{1/2-}(a,b)$ and $g_{T-} \in I_{T-}^{\beta}(L_2(0,T))$ for some $\beta > 1/2$. If the coefficient function a fulfills some regularity properties (see hypotheses $\boldsymbol{(H1)}$ and $\boldsymbol{(H2)}$ in Section 3.2) we prove in Theorem 3.5 a *uniform* local contraction principle on sufficiently small intervals $[t_0, t] \subset [0, T]$ in the space $W_{2,\infty}^{\beta'}(t_0, t)$ for some $1/2 < \beta' < \beta$. In [39] a non-uniform version was proved for almost all t_0 under the slightly weaker assumption $g \in W_2^{\beta}(0,T)$. This theorem can be extend to vector functions a_j in which case the integral operater is replaced by

$$Af := x_0 + \sum_{j=1}^{l} \int_{t_0}^{(\cdot)} a_j(f, \varphi)\, dg^j,$$

where the definition of the integrals is coordinatewise (Theorem 3.6).
If a function f on $[0,T]$ belongs to $W_{2,\infty}^{\beta}(t_0, t_1)$ for all small intervals $[t_0, t_1] \subset [0,T]$ such that $t_1 - t_0$ is less than a given constant Δ, then f is an element of $W_{2,\infty}^{\beta}(0,T)$ (Proposition 3.8). This allows a gluing procedure for obtaining a global solution of the (deterministic) differential equation on $\mathbb{R}^n$:

$$\begin{aligned} dx(t) &= \sum_{j=1}^{l} a_j(x(t), \varphi(t))\, dz^j(t) \\ x(0) &= x_0, \end{aligned}$$

where the real-valued driving functions z^j are such that $z_{T-}^j \in I_{T-}^{\beta}(L_2(0,T))$ for some $\beta > 1/2$. The parameter function φ is $\mathbb{R}^k$-valued and the coordinate functions are elements of $W_{2,\infty}^{1/2-}(0,T)$. If the vector-valued coefficient functions a_j fulfill the regularity condition hypotheses $\boldsymbol{(H1)}$ or $\boldsymbol{(H2)}$ we prove existence and uniqueness of a global solution $x \in W_{2,\infty}^{\beta-}(0,T)$ (Theorem 3.9). It can be obtained by Picard's iteration method as a consequence of the uniform local contraction principle and the gluing procedure.
For Hölder continuous driving functions z^j with Hölder exponent larger than

1/2 and without the parameter function φ the above differential equation was analyzed by Nualart and Răşcanu [21] and in the onedimensional case by Ruzmaikina [30]. Lyons [15] proved existence and uniqueness in the case when the driving functions z^j have bounded p-variation.

At the beginning of Chapter 4 we recall the notion of the so called *generalized stochastic forward integral* introduced by Zähle in [39] and its relationships to the fractional calculus from Chapter 2. For simplicity we only deal with continuous real-valued stochastic processes Z as integrators (see [11] for the Banach space setting), where the integral of a continuous stochastic process Y with respect to Z is defined as

$$\int_0^t Y\,dZ := \lim_{\varepsilon\to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t Y(s) \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u}\,ds\,du$$

whenever the limit exists in the sense of *uniform convergence in probability*, (ucp) for short. For functions f and g as in the definition of the generalized Lebesgue-Stieltjes integral and $Y := f$ and $Z := g$ the above limit exists and agrees with the integral via fractional calculus. Further, we define as in [39] the *generalized quadratic variation process* for a stochastic process Z by

$$[Z](t) := \lim_{\varepsilon\to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t \frac{(Z_{t-}(s+u) - Z_{t-}(s))^2}{u}\,ds\,du$$

if the (ucp)-limit exists. These notions are extensions of the integral and the quadratic variation introduced by Russo and Vallois [25], [26] and [27], where the authors used the limit $u \to 0$ without averaging the outer integral. If the stochastic process Z is a semimartingale then the integral in the sense of Russo and Vallois and in the sense of Zähle exists and agree with the usual Itô integral. As for classical Itô calculus a corresponding Itô formula remains valid. The idea for the proof goes back to Föllmer [9] in the special case of Riemann-Stieltjes approach.

The main result of this chapter is a construction of a solution to the stochastic differential equation

$$\begin{aligned} dX(t) &= \sum_{j=0}^m a_j(X(t),t)\,dZ^j(t) + b(X(t),t)\,dt \\ X(0) &= X_0, \end{aligned}$$

where the process Z^0 admits a finite generalized quadratic variation $[Z^0]$ in the above sense, and $[Z^0]$ and $Z^1, ..., Z^m$ are processes with sample paths in $I^{\beta}_{T-}(L_2(0,T))$ for some $\beta > 1/2$. The corresponding integrals with respect to $[Z^0], Z^1, ..., Z^m$ are understood in the sense of the generalized Lebesgue-Stieltjes integral from Chapter 2 and the one with respect to Z^0 in the sense of the generalized stochastic forward integral. In order to prove existence and uniqueness of a global pathwise solution in $W^{1/2-}_{2,\infty}(0,T)$ (Theorem 4.21) we use ideas from [42] for deriving a higher-dimensional version of the Doss-Sussman approach (see [5], [33]). The solution can be represented in the form

$$X(t) = h(Y(t), Z^0(t), t).$$

Here, the function h fulfills the auxiliary ordinary differential equation (ODE)

$$\begin{aligned} &\frac{\partial h}{\partial z}(y, z, t) = a_0(h(y, z, t), t) \\ &h(y, Z_0, t) = y \end{aligned}$$

and the process Y is a solution of an auxiliary stochastic differential equation of the form treated in Chapter 3.

In [28] the special case was treated, where $n = m = 1$, the coefficient functions are deterministic, Z^0 has finite quadratic variation in their sense, and Z^1 is a process of locally bounded variation. In this case fractional calculus is not needed. If the stochastic forward integral is replaced by the Skorohod integral or is determined by means of Wick products then one obtains different solutions (see Biagini et al. [3]).

In the last section of Chapter 4 we give an example where the ordinary quadratic variation (in the Riemann-Stieltjes or the Russo-Vallois sense) does not exist because of high oscillations, whereas the above average convergence holds true. The classical *Weierstrass-type functions* W or more general self-affine functions and their stochastic versions with scaling exponent $1/2$ are such examples. This can be shown with the methods from [37] and [41].

Chapter 5 is devoted first to the Hölder regularity of the solution of the fractional ODE from Chapter 3 where the integrator functions g^j are assumed to be

Hölder continuous of order $\beta > 1/2$. The condition that $g^j_{T-} \in I^{\beta}_{T-}(L_2(0,T))$ for existence and uniqueness of a solution implies only Hölder continuity of order $\beta - 1/2$ for the integrator functions g^j. Under an additional assumption on such integrators g^j Nualart and Răşcanu proved in [21] existence and uniqueness of a global solution in the space of Hölder continuous functions of order β for slightly different conditions on the coefficient functions a_j than in Chapter 3. Instead of the fractional Sobolev spaces these authors worked with modified versions using L_1 and L_∞ instead of L_2, which allows a global contraction principle based on a priori estimates. So, they don't need a local approach. Their method can be applied pathwise for stochastic differential equations driven by fractional Brownian motions B^H with Hurst exponents $H > 1/2$. Applying our contraction principle from Chapter 3 and an auxiliary proposition we derive that the solution of the ODE has without further assumptions the same Hölder regularity as the noises (Theorem 5.3).
In the second part of Chapter 5 we consider the SDE from Chapter 4 under the conditions that Z^0 admits a generalized quadratic variation process $[Z^0]$ and has Hölder continuous sample paths with Hölder exponent $\theta \leq 1/2$ and is not Hölder continuous of some order larger than $1/2$. The real-valued stochastic processes $[Z^0], Z^1, ..., Z^m$ are assumed to be Hölder continuous of order $\beta > 1/2$ and the random coefficient functions $a_0, a_1, ..., a_m, b$ fulfill the same conditions as in Section 4.3. In this case the Doss-Sussman-type approach from Chapter 4 remains valid and the pathwise solution is of the same Hölder regularity as the process Z^0 (Theorem 5.5).
The case $m = 1$, Z^0 is a Brownian motion and $Z^1 = B^H$ is a fractional Brownian motion with Hurst parameter $H > 1/2$ was investigated by Guerra and Nualart [10] for less regular, but adapted coefficient functions. They combined the methods from [21] with Itô integration.

Finally, we extend in the last chapter our SDE from Chapter 4 by adding a sum related to a marked point process $((T_k)_{k\geq 1}, (\xi_k)_{k\geq 1})$ and an associated counting process $N = (N(t))_{t\geq 0}$, which has with probability 1 only finitely many jumps on each interval $[0, t]$. The equation than becomes

$$X(t) = X_0 + \sum_{j=0}^{m} \int_0^t a_j(X(s), s)\, dZ^j(s) + \int_0^t b(X(s), s)\, ds$$
$$+ \sum_{k=1}^{N(t)} q(T_k, X(T_k-), \xi_k).$$

The jump times T_k of the counting process N are assumed to be independent of $Z^0, Z^1, ..., Z^m$. For the random vector fields $a_0, a_1, ..., a_m, b$ we suppose the same conditions as in Chapter 4 and for q only joint measurability in all arguments. Following the approach for stochastic differential equations with finitely many discontinuities (see for example in the book of Applebaum [1] for Lévy processes) we construct piecewise a pathwise global solution to this SDE with the same method as before.
For $m = 1$, Z^0 is a Brownian motion, $Z^1 = B^H$ is a fractional Brownian motion with Hurst parameter $H > 1/2$ and the marked point process is a compound Poisson process this differential equation was considered by Shevchenko [32]. He used again the slightly different conditions from [21] and [10] for the coefficient functions $a_0, a_1, ..., a_m, b$.

In summary, this thesis is a further contribution to the diverse and not completed theory of stochastic differential equations. There are still some questions remain to be answered. For example, what can we say about existence and uniqueness for the solution of the stochastic differential equation from Chapter 4, when there is more than one driving function with generalized quadratic variation not equal to zero?

Chapter 2

Fractal integrals and derivatives

2.1 General notations

Let $n, m \in \mathbb{N} \setminus \{0\}$. All (deterministic) functions and stochastic processes that are considered in this thesis have values in $\mathbb{R}^n$.
We denote for the vector valued function f the real valued component functions by $f^i, i = 1, ..., n$, i.e. $f(x) := (f^1(x), \dots, f^n(x))^T$.
The norm for real $(n \times m)$-matrices $A = (a_{i,j})$ is defined by

$$|A| := \left(\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{i,j}|^2 \right)^{1/2}.$$

Finite positive constants whose values are not of importance are denoted by C, their values may differ from one occurence to the other.

2.2 Fractional calculus

In this section we recall some important facts and results for fractional calculus. All proofs can be found in the book of Samko, Kilbas and Marichev [31].
Let us consider a finite interval $[a, b] \subset \mathbb{R}$. For $1 \leq p < \infty$ let $L_p(a, b)$ be the

space of functions $f : [a, b] \to \mathbb{R}^n$ such that

$$\|f\|_{L_p(a,b)} = \Big(\int_a^b |f(x)|^p \, dx\Big)^{1/p} < \infty$$

and $L_\infty(a, b)$ the space of functions $f : [a, b] \to \mathbb{R}^n$ with

$$\|f\|_{L_\infty(a,b)} = \operatorname*{ess\,sup}_{x\in[a,b]} |f(x)|.$$

If we consider instead of a compact interval $[a, b]$ the whole space $\mathbb{R}$ we use the notation $L_p(\mathbb{R})$.

The formula for an n-fold iterated integral reads as follows

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} f(x_n) \, dx_n \, dx_{n-1} \dots \, dx_1 = \frac{1}{(n-1)!} \int_a^x \frac{f(t)}{(x-t)^{1-n}} \, dt.$$

Because of the relation $(n-1)! = \Gamma(n)$, where Γ denotes the Gamma function, one could extend the right-hand side of the above equation with meaning for non-integer values of n.

Definition 2.1. Let $f \in L_1(a, b)$ and $\alpha > 0$ the *left-sided* and *right-sided fractional Riemann-Liouville integrals of* f *of order* α are defined for almost all $x \in (a, b)$ by

$$I^\alpha_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(y)}{(x-y)^{1-\alpha}} \, dy$$

and

$$I^\alpha_{b-} f(x) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(y)}{(y-x)^{1-\alpha}} \, dy.$$

For $\alpha > 0$ the term $(-1)^{-\alpha}$ is understood in the sense of choosing the main branch of the analytic function $z^\alpha, z \in \mathbb{C}$, with the cut along the positive half axis.

Let $\alpha, \beta > 0$ and $f \in L_1(a, b)$ then we have a *first composition formula*

$$I^\alpha_{\substack{a+\\(b-)}} \left(I^\beta_{\substack{a+\\(b-)}} f\right) = I^{\alpha+\beta}_{\substack{a+\\(b-)}} f.$$

Also the *first integration-by-parts formula* is valid

$$\int_a^b f(x) I^\alpha_{a+} g(x) \, dx = \int_a^b g(x) I^\alpha_{b-} f(x) \, dx,$$

if $f \in L_p(a,b), g \in L_q(a,b), p \geq 1, q \geq 1, \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ with $p > 1$ and $q > 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$.

The class of functions f which are representable as I^α_{a+} (resp. I^α_{b-})-integrals of some L_p-function φ $(p \geq 1)$ is denoted by $I^\alpha_{a+}(L_p(a,b))$ (resp. $I^\alpha_{b-}(L_p(a,b))$). For $0 < \alpha < 1$ the function φ agrees at almost all $x \in (a,b)$ with the *Weyl-Marchaud derivative of f of order α*

$$D^\alpha_{a+} f(x) := \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}}\, dy \right)$$

and

$$D^\alpha_{b-} f(x) := \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}}\, dy \right),$$

where the convergence of the integrals at the singularity $y = x$ holds pointwise for almost all x if $p = 1$ and in the L_p-sense if $p > 1$.

Note that in [31] and most of the literature the factor $(-1)^\alpha$ is usually omitted. The sets $I^\alpha_{\substack{a+\\(b-)}}(L_p(a,b))$ become Banach spaces by the norms

$$\|f\|_{I^\alpha_{\substack{a+\\(b-)}}(L_p(a,b))} := \|f\|_{L_p(a,b)} + \|D^\alpha_{\substack{a+\\(b-)}} f\|_{L_p(a,b)}.$$

If $f \in I^{\alpha+\beta}_{\substack{a+\\(b-)}}(L_1(a,b)), \alpha \geq 0, \beta \geq 0, \alpha + \beta \leq 1$ then we have the *second composition formula*

$$D^\alpha_{\substack{a+\\(b-)}} \left(D^\beta_{\substack{a+\\(b-)}} f \right) = D^{\alpha+\beta}_{\substack{a+\\(b-)}} f$$

and the *second integration-by-parts formula*

$$(-1)^\alpha \int_a^b D^\alpha_{a+} f(x) g(x)\, dx = \int_a^b f(x) D^\alpha_{b-} g(x)\, dx,$$

provided that $f \in I^\alpha_{a+}(L_p(a,b)), g \in I^\alpha_{b-}(L_q(a,b)), 0 \leq \alpha \leq 1, p \geq 1, q \geq 1, \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$.

The following inversion formula holds for $f \in I^\alpha_{a+}(L_p)$:

$$I^\alpha_{a+}(D^\alpha_{a+} f) = f.$$

We also get

$$D^\alpha_{a+}(I^\alpha_{a+} f) = f,$$

which is valid for general $f \in L_1(a,b)$.

For functions that are defined on the whole real axis $f : \mathbb{R} \to \mathbb{R}$ the left-sided and right-sided fractional Riemann-Liouville integrals of f of order $\alpha > 0$ are defined by

$$I_+^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}}\, dy, \quad x \in \mathbb{R} \tag{2.1}$$

and

$$I_-^\alpha f(x) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}\, dy, \quad x \in \mathbb{R}. \tag{2.2}$$

The integral (2.1) can be written as a convolution:

$$\begin{aligned} I_+^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^\infty y_+^{\alpha-1} f(x-y)\, dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} f(x+y)\, dy, \end{aligned}$$

where

$$y_+^{\alpha-1} = \begin{cases} y^{\alpha-1}, & y \geq 0 \\ 0, & y < 0. \end{cases}$$

A similar formulation is also valid for (2.2). The fractional integrals $I^\alpha_{\underset{(-)}{+}}$ are defined for all functions $f \in L_p(\mathbb{R})$ with $1 \leq p < 1/\alpha$. Just as in the case for the interval one can introduce for $0 < \alpha < 1$ the left-sided and right-sided Weyl-Marchaud derivatives at $x \in \mathbb{R}$ by

$$D_+^\alpha f(x) := \frac{\alpha}{\Gamma(1-\alpha)} \left(\int_0^\infty \frac{f(x) - f(x-y)}{y^{\alpha+1}}\, dy \right)$$

and

$$D_-^\alpha f(x) := \frac{\alpha(-1)^\alpha}{\Gamma(1-\alpha)} \left(\int_0^\infty \frac{f(x) - f(x+y)}{y^{\alpha+1}}\, dy \right),$$

where the hypersingular integrals on the right hand side converge in $L_p(\mathbb{R})$ if $f = I^\alpha_{\underset{(-)}{+}} \varphi$ for some $\varphi \in L_p(\mathbb{R})$ and $1 \leq p < 1/\alpha$. Here again the factor $(-1)^\alpha$ is omitted in [31].

If the function f is restricted to a interval $[a,b]$ then the relation

$$D^\alpha_{\underset{(b-)}{a+}} f(x) = \mathbf{1}_{(a,b)}(x) D^\alpha_{\underset{(-)}{+}} f(x)$$

holds.

2.3 Sobolev spaces

In the following we will introduce the function spaces that are needed in the later chapters. Throughout this section we just consider functions $f : [a, b] \to \mathbb{R}^n$ with $[a, b] \subset \mathbb{R}$ and $n \geq 1$.

The space of Hölder continuous functions of order $\lambda > 0$ is given by

$$H^\lambda(a, b) := \{f : [a, b] \to \mathbb{R}^n | \quad \|f\|_\lambda < \infty\},$$

where

$$\|f\|_\lambda := \sup_{t\in[a,b]} |f(t)| + \sup_{s,t\in[a,b]} \frac{|f(t) - f(s)|}{(t - s)^\lambda}.$$

Note that for $\mu > \lambda$ we have the embedding $H^\mu(a, b) \hookrightarrow H^\lambda(a, b)$.

Let $0 < \alpha < 1$. We define the space $\widetilde{W_2^\alpha}(a, b)$ by the seminorm

$$\|f\|_{\widetilde{W_2^\alpha}(a,b)} := \left(\int_a^b \int_a^b \frac{|f(y) - f(x)|^2}{|y - x|^{2\alpha+1}}\, dx\, dy\right)^{1/2}.$$

For $\beta > \alpha$ we have the embeddings $\widetilde{W_2^\beta}(a, b) \hookrightarrow \widetilde{W_2^\alpha}(a, b)$ and also $H^\beta(a, b) \hookrightarrow \widetilde{W_2^\alpha}(a, b)$. By adding a L_p-norm we obtain the fractional Sobolev (or Slobodeckij) spaces $W_2^\alpha(a, b)$ and $W_{2,\infty}^\alpha(a, b)$ given by the norms:

$$\|f\|_{W_2^\alpha(a,b)} := \|f\|_{L_2(a,b)} + \|f\|_{\widetilde{W_2^\alpha}(a,b)}$$

and

$$\|f\|_{W_{2,\infty}^\alpha(a,b)} := \|f\|_{L_\infty(a,b)} + \|f\|_{\widetilde{W_2^\alpha}(a,b)}.$$

Note that the Sobolev space $W_2^\alpha(a, b)$ coincides (up to norm equivalence) with the Besov space $B_{2,2}^\alpha(a, b)$ (see [34]).

Furthermore we also introduce the space $W_2^\alpha(b-)(a, b)$ by the norm

$$\|f\|_{W_2^\alpha(b-)(a,b)} := \left(\int_a^b \frac{|f(x)|^2}{|b - x|^{2\alpha}}\, dx\right)^{1/2} + \|f\|_{\widetilde{W_2^\alpha}(a,b)}.$$

Finally we denote,

$$W_{2,\infty}^{\beta-}(a, b) := \bigcap_{\alpha<\beta} W_{2,\infty}^\alpha(a, b).$$

In the next proposition we just want to state some relations between the above spaces and the spaces introduced in the last section.

Proposition 2.2. *(i) If* $\alpha p > 1$, *then* $I^{\alpha}_{\substack{+\\(-)}}(L_p(\mathbb{R})) \hookrightarrow H^{\lambda}(a,b)$, *for any* $0 < \lambda \leq \alpha - 1/p$ *(see [31], Theorem 3.6).*

(ii) For $\delta > 0$, $W_2^{\alpha+\delta}(b-)(a,b) \hookrightarrow I^{\alpha}_{b-}(L_2(a,b))$ *(see [39], Theorem 1.1 (ii)).*

(iii) Let $0 < \delta < 1 - \alpha$, *then* $I^{\alpha+\delta}_{\substack{a+\\(b-)}}(L_2(a,b)) \hookrightarrow \widetilde{W}_2^{\alpha}(a,b)$ *(see [8], Theorem 27).*

(iv) $g \in \widetilde{W}_2^{\alpha}(a,b)$ *implies* $g_{y-} \in W_2^{\alpha}(y-)(x,y)$ *for any* $x \in [a,b]$ *and Lebesgue almost all* $y \in (x,b)$ *(see [39], Theorem 1.1 (iv)).*

Later the following connection between Lipschitz mappings and the fractional Sobolev space $\widetilde{W}_2^{\alpha}(a,b)$ is needed.

Proposition 2.3. *Let* $f : [a,b] \to \mathbb{R}^n$ *and* $g : [a,b] \to \mathbb{R}^m$ *be two vector functions such that* $f \in \widetilde{W}_2^{\alpha_1}(a,b)$ *and* $g \in \widetilde{W}_2^{\alpha_2}(a,b)$ *and* $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ *a mapping which is Lipschitz continuous in both arguments uniformly with respect to the other argument. Then the function* $h(f(\cdot), g(\cdot)) : [a,b] \to \mathbb{R}^p$ *is an element of the space* $\widetilde{W}_2^{\min(\alpha_1,\alpha_2)}(a,b)$.

Proof. We just have to show that the seminorm

$$\|h(f,g)\|_{\widetilde{W}_2^{\min(\alpha_1,\alpha_2)}(a,b)} := \left(\int_a^b \int_a^b \frac{|h(f(y),g(y)) - h(f(x),g(x))|^2}{|y-x|^{2\min(\alpha_1,\alpha_2)+1}}\, dx\, dy \right)^{1/2}.$$

is finite. Because of the Lipschitz continuity of h we have

$$\begin{aligned}
&|h(f(y),g(y)) - h(f(x),g(x))| \\
&= |h(f(y),g(y)) - h(f(x),g(y)) + h(f(x),g(y)) - h(f(x),g(x))| \\
&\leq |h(f(y),g(y)) - h(f(x),g(y))| + |h(f(x),g(y)) - h(f(x),g(x))| \\
&\leq C\big(|f(y) - f(x)| + |g(y) - g(x)|\big).
\end{aligned}$$

This implies that

$$\begin{aligned}
\|h(f,g)\|_{\widetilde{W}_2^{\min(\alpha_1,\alpha_2)}(a,b)} \leq C &\left(\int_a^b \int_a^b \frac{|f(y) - f(x)|^2}{|y-x|^{2\min(\alpha_1,\alpha_2)+1}}\, dx\, dy \right)^{1/2} \\
&+ \left(\int_a^b \int_a^b \frac{|g(y) - g(x)|^2}{|y-x|^{2\min(\alpha_1,\alpha_2)+1}}\, dx\, dy \right)^{1/2}
\end{aligned}$$

$$= C\big(\|f\|_{\widetilde{W}_2^{\min(\alpha_1,\alpha_2)}(a,b)} + \|g\|_{\widetilde{W}_2^{\min(\alpha,\beta)}(a,b)}\big)$$
$$< \infty,$$

which completes the proof. □

Corollary 2.4. *Let $f : [a,b] \to \mathbb{R}^n$ be a vector functions such that $f \in \widetilde{W}_2^\alpha(a,b)$ and $h : \mathbb{R}^n \times [a,b] \to \mathbb{R}^p$ a mapping which is Lipschitz continuous in the first argument and Hölder continuous of order $\lambda > 0$ in the second argument always uniformly with respect to the other argument. Then the function $h(f(\cdot),\cdot) : [a,b] \to \mathbb{R}^p$ is an element of the space $\widetilde{W}_2^\beta(a,b)$ for all $0 < \beta < \min(\alpha,\lambda)$.*

2.4 Generalized Lebesgue-Stieltjes integrals

For a function $f : [a,b] \to \mathbb{R}$ we denote $f_{a+}(x) = \mathbf{1}_{(a,b)}(x)(f(x) - f(a+))$ and $f_{b-}(x) = \mathbf{1}_{(a,b)}(x)(f(x) - f(b-))$, assuming that the one-sided limits of f at the interval ends exist. Zähle [38] introduced the definition of *generalized Lebesgue-Stieltjes integral* with respect to irregular functions.

Definition 2.5. Suppose that f and g are functions such that $f(a+), g(a+)$ and $g(b-)$ exist, $f_{a+} \in I_{a+}^\alpha(L_p(a,b)), g_{b-} \in I_{b-}^{1-\alpha}(L_q(a,b))$ for some $p, q \geq 1, \frac{1}{p} + \frac{1}{q} \leq 1$ and $0 \leq \alpha \leq 1$. Then the generalized Lebesgue-Stieltjes integral of f with respect to g is defined by

$$\int_a^b f(x)\,dg(x) = (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x)\,dx + f(a+)(g(b-) - g(a+)). \tag{2.3}$$

Because of the choice of the left-sided derivatives of f and right-sided derivatives of g this integral is directed and also sometimes called the *forward integral* of f with respect to g.

Remark 2.6. (i) It was also proven in [38] that this definition is correct, i.e. the definition of the integral is independent of the choice of α.

(ii) If $\alpha p < 1$ this integral equals

$$\int_a^b f\,dg = (-1)^\alpha \int_a^b D^\alpha_{a+} f(x) D^{1-\alpha}_{b-} g_{b-}(x)\,dx, \tag{2.4}$$

which is determined for general $f \in I^\alpha_{a+}(L_p(a,b))$ with $\limsup_{x\to a} f(x) < \infty$.

The above defined integral has some nice properties. For instance it is an additive function of the boundary.

Theorem 2.7. *([38], Thm. 2.5) Let $a \le c < d < e \le b$.*

(i) One has that

$$\int_c^d f\,dg = \int_a^b \mathbf{1}_{(d,c)} f\,dg,$$

if both integrals exist in the sense of Definition 2.5.

(ii) One has that

$$\int_c^d f\,dg + \int_d^e f\,dg = \int_c^e f\,dg - f(d)(g(d+) - g(d-)),$$

if all summands are determinded as in Definition 2.5.

Now we want to compare the generalized Lebesgue-Stieltjes integral with the the Riemann-Stieltjes integral for the special case where f and g are Hölder continuous of summed order greater than 1. Le us briefly recall the definition of the the Riemann-Stieltjes integral. Namely,

$$(RS) - \int_a^b f\,dg := \lim_{\Delta\to 0} \sum_i f(x_i^*)(g(x_{i+1}) - g(x_i)) \tag{2.5}$$

and the convergence holds uniformly for all finite partitions $a = x_0 \le x_1 \le \dots x_{n+1} = b$ with $x_i \le x_i^* \le x_{i+1}$ and $\sup_i |x_{i+1} - x_i| < \Delta, i = 0,\dots,n$. Theorem 4.2.1 of [38] states that if $f \in H^\lambda(a,b)$ and $g \in H^\mu(a,b)$ for some $\lambda + \mu > 1$. Then the Riemann-Stieltjes integral $(RS) - \int_a^b f\,dg$ in the sense of (2.5) exists (see Young [35]) and agrees with the generalized Lebesgue-Stieltjes integral $\int_a^b f\,dg$ defined by (2.3). Moreover the integral as a function of the upper or lower boundary is Hölder continuous of order μ, i.e.

$$\mathbf{1}_{(a,b)} \int_a^{(\cdot)} f\,dg \in H^\mu(a,b) \quad \text{and} \quad \mathbf{1}_{(a,b)} \int_{(\cdot)}^b f\,dg \in H^\mu(a,b) \tag{2.6}$$

(see [38], Proposition 4.4.1).

We also have some estimates for the norms of the Sobolev spaces introduced in Section 2.2 for the generalized Lebesgue-Stieltjes integral as a function of the upper boundary. The following theorem is essential for the *contraction property* in the next chapter.

Theorem 2.8. *Let* $f : [a,b] \to \mathbb{R}^n$ *and* $g : [a,b] \to \mathbb{R}$. *Suppose* $0 < \alpha < 1/2$ *and* $0 < \beta < 1$. *Then we have*

(i)

$$\Big\| \int_a^{(\cdot)} f\,dg \Big\|_{W^\beta_{2,\infty}(a,b)} \le C\Big(\|f\|_{W^{\max(\alpha,\alpha+\beta-1/2)}_{2,\infty}(a,b)} \|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2(a,b))} + \|f\|_{L_\infty(a,b)} \|g\|_{\widetilde{W}^\beta_2(a,b)} \Big),$$

where the integral is defined coordinatewise in the sense of Definition 2.5,

(ii)

$$\Big\| \int_a^{(\cdot)} f\,dg \Big\|_{W^\beta_{2,\infty}(a,b)} \le C \|f\|_{W^\beta_{2,\infty}(a,b)} \|g_{b-}\|_{W^\beta_2(b-)(a,b)},$$

if $\beta > 1/2$ *and the integral is defined in the same way as in (i).*

Proof. Let $f^i, i = 1, ..., n$ be the real valued component functions of f, i.e. $f(x) := (f^1(x), \dots, f^n(x))^T$. The supremum norm is given by

$$\Big\| \int_a^{(\cdot)} f\,dg \Big\|_{L_\infty(a,b)} = \sup_{x\in[a,b]} \Big(\sum_{i=1}^n \Big| \int_a^x f^i\,dg \Big|^2 \Big)^{1/2}.$$

For the component functions f^i we use the estimates

$$\Big| \int_a^x f^i\,dg \Big| \le C \|f^i\|_{W^\alpha_{2,\infty}(a,b)} \|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2(a,b))}$$

(see [39], Thm. 1.3) and hence we can conclude that

$$\begin{aligned}\left\|\int_a^{(\cdot)} f\,dg\right\|_{L_\infty(a,b)} &\le \left(\sum_{i=1}^n C\|f^i\|^2_{W^\alpha_{2,\infty}(a,b)}\|g_{b-}\|^2_{I^{1-\alpha}_{b-}(L_2(a,b))}\right)^{1/2}\\ &\le C\|f\|_{W^\alpha_{2,\infty}(a,b)}\|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2(a,b))}.\end{aligned}$$

The estimate for the $\widetilde{W}^\alpha_2(a,b)$-seminorm is

$$\begin{aligned}\left\|\int_a^{(\cdot)} f\,dg\right\|_{\widetilde{W}^\alpha_2(a,b)} &= \left(\int_a^b\int_a^b \frac{\sum_{i=1}^n |\int_a^y f^i\,dg - \int_a^x f^i\,dg|^2}{|y-x|^{2\beta+1}}\,dx\,dy\right)^{1/2}\\ &\le \left(\int_a^b\int_a^b \frac{\left(\sum_{i=1}^n |\int_a^y f^i\,dg - \int_a^x f^i\,dg|\right)^2}{|y-x|^{2\beta+1}}\,dx\,dy\right)^{1/2}\\ &\le \sum_{i=1}^n\left(\int_a^b\int_a^b \frac{|\int_a^y f^i\,dg - \int_a^x f^i\,dg|^2}{|y-x|^{2\beta+1}}\,dx\,dy\right)^{1/2},\end{aligned}$$

where we used the triangle inequality. Finally we apply again the estimates for the component functions f^i

$$\begin{aligned}&\left(\int_a^b\int_a^b \frac{|\int_a^y f^i\,dg - \int_a^x f^i\,dg|^2}{|y-x|^{2\beta+1}}\,dx\,dy\right)^{1/2}\\ &\le C\Big(\|f^i\|_{W^{\max(\alpha,\alpha+\beta-1/2)}_{2,\infty}(a,b)}\|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2(a,b))} + \|f^i\|_{L_\infty(a,b)}\|g\|_{\widetilde{W}^\beta_2(a,b)}\Big)\end{aligned}$$

(see [39], Thm. 1.3) and we obtain

$$\begin{aligned}&\left\|\int_a^{(\cdot)} f\,dg\right\|_{\widetilde{W}^\alpha_2(a,b)}\\ &\le \sum_{i=1}^n C\Big(\|f^i\|_{W^{\max(\alpha,\alpha+\beta-1/2)}_{2,\infty}(a,b)}\|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2(a,b))} + \|f^i\|_{L_\infty(a,b)}\|g\|_{\widetilde{W}^\beta_2(a,b)}\Big)\\ &\le C\Big(\|f\|_{W^{\max(\alpha,\alpha+\beta-1/2)}_{2,\infty}(a,b)}\|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2(a,b))} + \|f\|_{L_\infty(a,b)}\|g\|_{\widetilde{W}^\beta_2(a,b)}\Big).\end{aligned}$$

The proof of (i) is thus complete. The statement of (ii) follows from (i) if we choose α such that $1-\beta<\alpha<1/2$ using in addition the estimate

$$\|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2(a,b))} \le C\|g_{b-}\|_{W^\beta_2(b-)(a,b)}.$$

Again we refer to Theorem 1.3 in [39]. □

Remark 2.9. The constant C in Theorem 2.8 tends to infinity in (i) as $\alpha \nearrow 1/2$ and also in (ii) as $\beta \searrow 1/2$.

2.5 Fractional Brownian motion

This section is devoted to the stochastic process called *fractional Brownian motion* (fBm). It was first introduced in 1940 by Kolmogorov [14] in a Hilbert space framework. He used at that time the designation *Wiener Helix*. In 1951 Hurst used this process to describe the long-term storage capacity in reservoirs (see [12]). The parameter H is in honour of Hurst called *Hurst parameter*. The name fractional Brownian motion goes back to the paper of Mandelbrot and Van Ness in 1968 [16]. They provided a stochastic integral representation of this process in terms of a standard Brownian motion. Because of the long-range dependence property of the fBm it is a suitable process for models in finance, climate and medical image processing.
For further details to the following results we refer to [3] and [20]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 2.10. A fractional Brownian motion with Hurst parameter $H \in (0, 1)$ is a Gaussian process $(B^H(t))_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with the properties

(i) $B^H(0) = 0$,

(ii) $\mathbb{E}\big[B^H(t)\big] = 0, \quad t \geq 0$,

(iii) $\mathbb{E}\big[B^H(t)B^H(s)\big] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad s, t \geq 0$.

Because of the relation $\mathbb{E}(B^H(t) - B^H(s))^2 = |t-s|^{2H}$ and the fact that B^H is a Gaussian process we can apply the Kolmogorov theorem (see e.g. [1]) to show that the fractional Brownian motion has a continuous modification.

If $H = 1$, fractional Brownian motion can be constructed as follows:

$$B^1(t) = t\xi, \quad t \geq 0,$$

where ξ is a standard normal random variable.

For the special case $H = 1/2$, the fBm is a standard Brownian motion.

The covarinance function of the fBm with Hurst parameter H is homogeneous of order $2H$. Hence B^H is a H-self-similar process, i.e. for any constant $c > 0$ we get

$$Law(B^H(ct), t \geq 0) = Law(c^H B^H(t), t \geq 0).$$

Moreover the process B^H admits a version for which the sample paths are almost surely Hölder continuous of order α for all $0 < \alpha < H$.

It is known that the increments of the standard Brownian motion are independent. This is not true for fractional Brownian motion with Hurst parameter $H \neq 1/2$. The increments for B^H are negatively correlated if $H < 1/2$ and positively correlated if $H > 1/2$.

The autocovariance functions $\rho(n) = Cov(X(k), X(k+n))$ for a stationary sequence $(X(n))_{n\in\mathbb{N}}$ exhibits long-range dependence if

$$\lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1$$

for some constant $c \in (0, \infty)$ and $\alpha \in (0, 1)$. So the dependence between $X(k)$ and $X(k+n)$ decays slowly for n going to ∞ and we have

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

Choosing $X(k) := B^H(k) - B^H(k-1)$ and $X(k+n) := B^H(k+n) - B^H(k+n-1)$ we obtain the long-range property if and only if $H > 1/2$, because

$$\rho_H(n) = \frac{1}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}] \sim H(2H-1)n^{2H-2},$$

as n tends to infinity.

A last property of fractional Brownian motion is that it is only a semimartingale for $H = 1/2$. So the classical stochastic integration in the sense of Itô does not work. But using the Hölder continuity of the sample paths of B^H we obtain the pathwise existence of the integral

$$\int_0^t f\, dB^H, \quad 0 < t \leq T,$$

in the sense of (2.3) with probability 1 for any measurable random function f on the finite interval $[0, T] \subset \mathbb{R}$ such that $f_{0+} \in I_{0+}^{\alpha}(L_1(0, T))$ with probability

1 with $\alpha > 1 - H$. There is no assumption of adaptedness needed. For a detailed survey about integration with respect to fBm we recommend [22].

Chapter 3

Differential equations driven by functions with fractional smoothness of order greater than 1/2

3.1 The problem

In this chapter we want to consider the deterministic differential equation in $\mathbb{R}^n$:

$$\begin{aligned} dx(t) &= \sum_{j=1}^{m} a_j(x(t), \varphi(t))\, dz^j(t) \\ x(0) &= x_0 \end{aligned} \tag{3.1}$$

for $t \in [0,T], x_0 \in \mathbb{R}^n, n \geq 1$ and real-valued driving functions $z^1, \ldots, z^m$ such that $z^j_{T-} \in I^{\beta}_{T-}(L_2(0,T))$, for some $1/2 < \beta < 1$. The parameter function φ takes values in $\mathbb{R}^k, k \geq 1$ with coordinate functions in $W^{1/2-}_{2,\infty}(0,T)$ and the a_j are $\mathbb{R}^n$-valued vector fields on $\mathbb{R}^n \times \mathbb{R}^k$. The equation becomes precise via integration. Namely a solution satisfies

$$x(t) = x_0 + \sum_{j=1}^{m} \int_0^t a_j(x(s), \varphi(s))\, dz^j(s), \tag{3.2}$$

where the integrals may be interpreted coordinatewise in the sense of (2.3). We are looking for a global $\mathbb{R}^n$-valued solution of (3.1) in the space $W_{2,\infty}^{\beta-}(0,T)$. To this aim we have to assume some regularity conditions for the vector-valued coefficient functions a_j (see hypotheses **(*H*1)** and **(*H*2)** below). The result about the existence and uniqueness of such a solution follows by Picard's iteration method, which can be applied because of a uniform local contraction principle explained in the next section.
If we exchange the deterministic driving functions $z^1, \dots, z^m$ by stochastic processes $Z^1, \dots, Z^m$ with sample paths in $I_{T-}^{\beta}(L_2(0,T))$, for example fractional Brownian motions, then we obtain an auxiliary stochastic differential equation. This set-up will be treated in Chapter 4.

3.2 Integral operator and contraction property

Let us fix a real-valued integrator function g with $g_{T-} \in I_{T-}^{\beta}(L_2(0,T))$ for some $1/2 < \beta < 1$. We also consider a vector-valued *parameter function* $\varphi : [0,T] \to \mathbb{R}^k$ with $\varphi \in W_{2,\infty}^{1/2-}(0,T)$ and a transformation mapping $a : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$. For vector functions $f : [0,T] \to \mathbb{R}^n$ with $f \in W_{2,\infty}^{\beta'}(0,T)$, where $1/2 < \beta' < \beta < 1$, we introduce for fixed $x_0 \in \mathbb{R}^n$ the non-linear integral operator A by

$$Af := x_0 + \int_0^{(\cdot)} a(f, \varphi)\, dg \tag{3.3}$$

with coordinatewise definition of the integral.
If the transformation mapping a fullfills one of the hypotheses **(*H*1)** or **(*H*2)** below we infer a certain contraction property for the integral operator. This is the essential tool for solving the differential equation (3.1).

$(\boldsymbol{H1})$ The function $a(x,\tau)$ is differentiable in the first variable and there exist some constants $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3 \in (0,\infty)$ such that:

$$\begin{aligned}
\Big|\frac{\partial a}{\partial x}(x,\tau)\Big| &\leq \mathcal{L}_0, \\
\Big|\frac{\partial a}{\partial x}(x,\tau) - \frac{\partial a}{\partial x}(y,\tau)\Big| &\leq \mathcal{L}_1|x-y|, \\
\Big|\frac{\partial a}{\partial x}(x,\tau) - \frac{\partial a}{\partial x}(x,\sigma)\Big| &\leq \mathcal{L}_2|\tau-\sigma|, \\
\Big|a(x,\tau) - a(x,\sigma)\Big| &\leq \mathcal{L}_3|\tau-\sigma|,
\end{aligned}$$

for all $x, y \in \mathbb{R}^n$ and $\tau, \sigma \in \mathbb{R}^k$.

$(\boldsymbol{H2})$ The function $a(x,\tau)$ is differentiable in both variables and there exist some constants $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_4 \in (0,\infty)$ such that:

$$\begin{aligned}
\Big|\frac{\partial a}{\partial x}(x,\tau)\Big| &\leq \mathcal{L}_0, \\
\Big|\frac{\partial a}{\partial x}(x,\tau) - \frac{\partial a}{\partial x}(y,\tau)\Big| &\leq \mathcal{L}_1|x-y|, \\
\Big|\frac{\partial a}{\partial \tau}(x,\tau) - \frac{\partial a}{\partial \tau}(y,\tau)\Big| &\leq \mathcal{L}_2|x-y|, \\
\Big|\frac{\partial a}{\partial \tau}(x,\tau)\Big| &\leq \mathcal{L}_4,
\end{aligned}$$

for all $x, y \in \mathbb{R}^n$ and $\tau, \sigma \in \mathbb{R}^k$.

In a first step we study the behaviour of the mapping

$$f \mapsto a(f,\varphi).$$

Proposition 3.1. *Let f and h be $\mathbb{R}^n$-valued vector functions and φ be a $\mathbb{R}^k$-valued parameter function with $f, h, \varphi \in W^{\mu}_{2,\infty}(0,T)$, where $0 < \mu < 1$. The mapping a satisfies hypothesis* **$(\boldsymbol{H1})$** *or* **$(\boldsymbol{H2})$**. *Then*

$$\|a(f,\varphi) - a(h,\varphi)\|_{L_\infty(0,T)} \leq \mathcal{L}_0\|f-h\|_{L_\infty(0,T)}$$

and

$$\begin{aligned}\|a(f,\varphi)-a(h,\varphi)\|_{\widetilde{W}_2^{\mu}(0,T)} \leq \quad & \mathcal{L}_0\|f-h\|_{\widetilde{W}_2^{\mu}(0,T)} \\ & + \mathcal{L}_1\|f-h\|_{L_\infty(0,T)}\Big(\|f\|_{\widetilde{W}_2^{\mu}(0,T)}+\|h\|_{\widetilde{W}_2^{\mu}(0,T)}\Big) \\ & + \mathcal{L}_2\|f-h\|_{L_\infty(0,T)}\|\varphi\|_{\widetilde{W}_2^{\mu}(0,T)}.\end{aligned}$$

Proof. By the mean value theorem we get for the L_∞-norm the following estimate

$$\begin{aligned}\|a(f,\varphi)-a(h,\varphi)\|_{L_\infty(0,T)} &= \sup_{t\in[0,T]}|(a(f(t),\varphi(t))-a(h(t),\varphi(t)))| \\ &= \sup_{t\in[0,T]}\left|\int_0^1\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(t))(f(t)-h(t))\,d\theta\right| \\ &\leq \sup_{t\in[0,T]}\int_0^1\left|\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(t))\right||(f(t)-h(t))|\,d\theta \\ &\leq \mathcal{L}_0\sup_{t\in[0,T]}|f(t)-h(t)| \\ &= \mathcal{L}_0\|f-h\|_{L_\infty(0,T)}.\end{aligned}$$

Now we have to show that for both conditions $\boldsymbol{(H1)}$ and $\boldsymbol{(H2)}$ the inequality

$$\begin{aligned}&\big|a(f(t),\varphi(t))-a(h(t),\varphi(t))-a(f(s),\varphi(s))+a(f(s),\varphi(s))\big| \\ &\leq \quad \mathcal{L}_0|f(t)-f(s)-h(t)+h(s)| \\ &\quad + \|f-h\|_{L_\infty(0,T)}\Big(\mathcal{L}_2|\varphi(t)-\varphi(s)|+\mathcal{L}_1\big(|f(t)-f(s)|+|h(t)-h(s)|\big)\Big)\end{aligned}$$

is fulfilled.

Let us first consider hypothesis $\boldsymbol{(H1)}$. Here, we have that

$$\begin{aligned}&\big|a(f(t),\varphi(t))-a(h(t),\varphi(t))-a(f(s),\varphi(s))+a(h(s),\varphi(s))\big| \\ =&\bigg|\int_0^1\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(t))(f(t)-h(t))\,d\theta \\ &-\int_0^1\frac{\partial a}{\partial x}(\theta f(s)+(1-\theta)h(s),\varphi(s))(f(s)-h(s))\,d\theta\bigg|\end{aligned}$$

$$
\begin{aligned}
=&\Big|\int_0^1 \frac{\partial a}{\partial x}(\theta f(s)+(1-\theta)h(s),\varphi(s))(f(t)-f(s)-h(t)+h(s))\,d\theta\\
&+\int_0^1\Big(\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(t))-\frac{\partial a}{\partial x}(\theta f(s)+(1-\theta)h(s),\varphi(s))\Big)\\
&\quad(f(t)-h(t))\,d\theta\Big|\\
\leq&\int_0^1 |f(t)-f(s)-h(t)+h(s)|\Big|\frac{\partial a}{\partial x}(\theta f(s)+(1-\theta)h(s),\varphi(s))\Big|\,d\theta\\
&+\int_0^1 |f(t)-h(t)|\Big|\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(t))\\
&\qquad-\frac{\partial a}{\partial x}(\theta f(s)+(1-\theta)h(s),\varphi(s))\Big|\,d\theta\\
=:&S_1+S_2.
\end{aligned}
$$

The first summand can be estimated by

$$
S_1\leq \mathcal{L}_0|f(t)-f(s)-h(t)+h(s)|.
$$

For the second summand we get

$$
\begin{aligned}
S_2=&\int_0^1\Big|\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(t))-\frac{\partial a}{\partial x}(\theta f(s)+(1-\theta)h(s),\varphi(s))\Big|\,d\theta\\
&\quad|f(t)-h(t)|\\
=&\int_0^1\Big|\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(t))-\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(s))\\
&+\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(s))-\frac{\partial a}{\partial x}(\theta f(s)+(1-\theta)h(s),\varphi(s))\Big|\,d\theta\\
&\quad|f(t)-h(t)|\\
\leq&\int_0^1\Big|\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(t))-\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(s))\Big|\\
&+\Big|\frac{\partial a}{\partial x}(\theta f(t)+(1-\theta)h(t),\varphi(s))-\frac{\partial a}{\partial x}(\theta f(s)+(1-\theta)h(s),\varphi(s))\Big|\,d\theta\\
&\quad|f(t)-h(t)|\\
\leq&\int_0^1 \mathcal{L}_2|\varphi(t)-\varphi(s)|+\mathcal{L}_1\Big|\theta(f(t)-f(s))+(1-\theta)(h(t)-h(s))\Big|\,d\theta\\
&\quad|f(t)-h(t)|\\
\leq&\|f-h\|_{L_\infty(0,T)}\Big(\mathcal{L}_2|\varphi(t)-\varphi(s)|+\mathcal{L}_1\big(|f(t)-f(s)|+|h(t)-h(s)|\big)\Big).
\end{aligned}
$$

So the inequality is fullfilled for hypothesis $\boldsymbol{(H1)}$.

Now we assume hypothesis $\boldsymbol{(H2)}$. Here we use the decomposition

$$\begin{aligned}
&\big|a(f(t),\varphi(t)) - a(h(t),\varphi(t)) - a(f(s),\varphi(s)) + a(f(s),\varphi(s))\big| \\
&\leq \big|a(f(t),\varphi(t)) - a(f(s),\varphi(t)) - a(h(t),\varphi(t)) + a(h(s),\varphi(t))\big| \\
&\quad + \big|a(f(s),\varphi(t)) - a(f(s),\varphi(s)) - a(h(s),\varphi(t)) + a(h(s),\varphi(s))\big| \\
&=: S_3 + S_4.
\end{aligned}$$

The mean value theorem implies

$$\begin{aligned}
S_3 =& \Big|\int_0^1 \frac{\partial a}{\partial x}(\theta f(t) + (1-\theta)f(s),\varphi(t))(f(t)-f(s))\,d\theta \\
&- \int_0^1 \frac{\partial a}{\partial x}(\theta h(t) + (1-\theta)h(s),\varphi(t))(h(t)-h(s))\,d\theta\Big| \\
\leq& \Big|\int_0^1 \frac{\partial a}{\partial x}(\theta f(t) + (1-\theta)f(s),\varphi(t))(f(t)-f(s)-h(t)+h(s))\,d\theta\Big| \\
&+ \Big|\int_0^1 \Big(\frac{\partial a}{\partial x}(\theta h(t) + (1-\theta)h(s),\varphi(t)) - \frac{\partial a}{\partial x}(\theta f(t) + (1-\theta)f(s),\varphi(t))\Big) \\
&(h(t)-h(s))\,d\theta\Big| \\
\leq&\ \ \mathcal{L}_0|f(t)-f(s)-h(t)+h(s)| \\
&+ \mathcal{L}_1\big|\theta h(t) + (1-\theta)h(s) - \theta f(t) - (1-\theta)f(s)\big||h(t)-h(s)| \\
\leq&\ \ \mathcal{L}_0|f(t)-f(s)-h(t)+h(s)| \\
&+ \frac{1}{2}\mathcal{L}_1\big(|h(t)-f(t)| + |h(s)-f(s)|\big)|h(t)-h(s)| \\
\leq&\ \ \mathcal{L}_0|f(t)-f(s)-h(t)+h(s)| \\
&+ \mathcal{L}_1\|f-h\|_{L_\infty(0,T)}\big(|h(t)-h(s)| + |f(t)-f(s)|\big).
\end{aligned}$$

The summand S_4 can be estimated by

$$\begin{aligned}
S_4 =& \Big|\int_0^1 \frac{\partial a}{\partial \tau}(f(s),\theta\varphi(t) + (1-\theta)\varphi(s))(\varphi(t)-\varphi(s))\,d\theta \\
&- \int_0^1 (\varphi(t)-\varphi(s))\frac{\partial a}{\partial \tau}(h(s),\theta\varphi(t) + (1-\theta)\varphi(s))\,d\theta\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \left| \frac{\partial a}{\partial \tau}(f(s), \theta\varphi(t) + (1-\theta)\varphi(s)) - \frac{\partial a}{\partial \tau}(h(s), \theta\varphi(t) + (1-\theta)\varphi(s)) \right| d\theta \\
&\quad |\varphi(t) - \varphi(s)| \\
&\leq \mathcal{L}_2 |f(s) - h(s)| |\varphi(t) - \varphi(s)| \\
&\leq \mathcal{L}_2 \|f - h\|_{L_\infty(0,T)} |\varphi(t) - \varphi(s)|,
\end{aligned}
$$

and hence the inequality is also satisfied for hypothesis $\boldsymbol{(H2)}$.
Now we can conclude that

$$
\begin{aligned}
&\|a(f,\varphi) - a(h,\varphi)\|_{\widetilde{W}_2^\mu(0,T)} \\
=&\bigg(\int_0^T \int_0^t \big| a(f(t), \varphi(t)) - a(h(t), \varphi(t)) - a(f(s), \varphi(s)) + a(f(s), \varphi(s)) \big|^2 \\
&\qquad (t-s)^{-(2\mu+1)}\, ds\, dt \bigg)^{1/2} \\
\leq&\ \mathcal{L}_0 \bigg(\int_0^T \int_0^t |f(t) - f(s) - h(t) + h(s)|^2 (t-s)^{-(2\mu+1)}\, ds\, dt \bigg)^{1/2} \\
&+ \mathcal{L}_1 \|f-h\|_{L_\infty(0,T)} \bigg(\int_0^T \int_0^t |f(t) - f(s)|^2 (t-s)^{-(2\mu+1)}\, ds\, dt \bigg)^{1/2} \\
&+ \mathcal{L}_1 \|f-h\|_{L_\infty(0,T)} \bigg(\int_0^T \int_0^t |h(t) - h(s)|^2 (t-s)^{-(2\mu+1)}\, ds\, dt \bigg)^{1/2} \\
&+ \mathcal{L}_2 \|f-h\|_{L_\infty(0,T)} \bigg(\int_0^T \int_0^t |\varphi(t) - \varphi(s)|^2 (t-s)^{-(2\mu+1)}\, ds\, dt \bigg)^{1/2} \\
\leq&\ \mathcal{L}_0 \|f-h\|_{\widetilde{W}_2^\mu(0,T)} \\
&+ \mathcal{L}_1 \|f-h\|_{L_\infty(0,T)} \Big(\|f\|_{\widetilde{W}_2^\mu(0,T)} + \|h\|_{\widetilde{W}_2^\mu(0,T)} \Big) \\
&+ \mathcal{L}_2 \|f-h\|_{L_\infty(0,T)} \|\varphi\|_{\widetilde{W}_2^\mu(0,T)}.
\end{aligned}
$$

This completes the proof. □

Remark 3.2. If we just assume hypothesis $\boldsymbol{(H2)}$ the assertion of Proposition 3.1 can be changed to

$$
\begin{aligned}
\|a(f,\varphi) - a(h,\varphi)\|_{\widetilde{W}_2^\mu(0,T)} \leq&\ \mathcal{L}_0 \|f-h\|_{\widetilde{W}_2^\mu(0,T)} \\
&+ \mathcal{L}_1 \|f-h\|_{L_\infty(0,T)} \min\Big(\|f\|_{\widetilde{W}_2^\mu(0,T)}, \|h\|_{\widetilde{W}_2^\mu(0,T)} \Big) \\
&+ \mathcal{L}_2 \|f-h\|_{L_\infty(0,T)} \|\varphi\|_{\widetilde{W}_2^\mu(0,T)}
\end{aligned}
$$

and the rest remains unchanged. A proof for the one-dimensional case may be found in [39].

Theorem 3.3. *Let $1/2 < \beta' < \beta < 1$, f, h are $\mathbb{R}^n$-valued vector functions with $f, h \in W_{2,\infty}^{\beta'}(0,T)$. The parameter function φ takes values in $\mathbb{R}^k$ with $\varphi \in W_{2,\infty}^{1/2-}(0,T)$ and the real-valued function g is such that $g_{T-} \in I_{T-}^{\beta}(L_2(0,T))$. Then we get for any mapping a satisfying **(H1)** or **(H2)** and any $0 \le t_0 < t \le T$,*

$$\begin{aligned}
&\left\| \int_{t_0}^{(\cdot)} a(f,\varphi)\,dg - \int_{t_0}^{(\cdot)} a(h,\varphi)\,dg \right\|_{W_{2,\infty}^{\beta'}(t_0,t)} \\
\le& C\Big[\mathcal{L}_0 \|f-h\|_{W_{2,\infty}^{\beta'}(t_0,t)} + \mathcal{L}_1 \|f-h\|_{L_\infty(t_0,t)} \Big(\|f\|_{\widetilde{W}_2^{\beta'}(t_0,t)} + \|h\|_{\widetilde{W}_2^{\beta'}(t_0,t)} \Big) \\
&+ \mathcal{L}_2 \|f-h\|_{L_\infty(t_0,t)} \|\varphi\|_{W_{2,\infty}^{1/2-}(t_0,t)} \Big] \|g_{t-}\|_{W_2^{\gamma}(t-)(t_0,t)},
\end{aligned}$$

where $\frac{\beta+\beta'}{2} < \gamma < \beta$.

Proof. By Theorem 2.8 and Proposition 3.1 we have

$$\begin{aligned}
\left\| \int_{t_0}^{(\cdot)} a(f,\varphi)\,dg - \int_{t_0}^{(\cdot)} a(h,\varphi)\,dg \right\|_{W_{2,\infty}^{\beta'}(t_0,t)} \le& C\Big[\mathcal{L}_0 \|f-h\|_{W_{2,\infty}^{\max(\alpha,\alpha+\beta'-1/2)}(t_0,t)} \\
&+ \mathcal{L}_1 \|f-h\|_{L_\infty(t_0,t)} \Big(\|f\|_{\widetilde{W}_2^{\max(\alpha,\alpha+\beta'-1/2)}(t_0,t)} \\
&\quad + \|h\|_{\widetilde{W}_2^{\max(\alpha,\alpha+\beta'-1/2)}(t_0,t)} \Big) \\
&+ \mathcal{L}_2 \|f-h\|_{L_\infty(t_0,t)} \|\varphi\|_{W_{2,\infty}^{\max(\alpha,\alpha+\beta'-1/2)}} \Big] \|g_{t-}\|_{I_{t-}^{1-\alpha}(L_2(t_0,t))} \\
&+ \mathcal{L}_0 \|f-h\|_{L_\infty(t_0,t)} \|g\|_{\widetilde{W}_2^{\beta'}(t_0,t)}
\end{aligned}$$

for all $0 < \alpha < 1/2$. Choosing $\alpha := 1 - \beta + \frac{\beta-\beta'}{2}$ so that $\alpha < \alpha + \beta' - 1/2 < 1/2 < \beta'$ and using the estimates

$$\|g_{t-}\|_{I_{t-}^{1-\alpha}(L_2(t_0,t))} \le C \|g_{t-}\|_{W_2^{\gamma}(t-)(t_0,t)}$$

and

$$\|g\|_{\widetilde{W}_2^{\beta'}(t_0,t)} \le C \|g_{t-}\|_{W_2^{\gamma}(t-)(t_0,t)}$$

(see Proposition 2.2) the proof is completed. □

Lemma 3.4. *Let $g_{T-} \in I^{\beta}_{T-}(L_2(0,T))$ and $1/2 < \gamma < \beta < 1$. Then we get*

$$\lim_{\Delta \to 0} \sup_{t_0 \in [0,T]} \|g_{(t_0+\Delta)-}\|_{W_2^{\gamma}((t_0+\Delta)-)(t_0,t_0+\Delta)} = 0.$$

Proof. By definition,

$$\begin{aligned}\|g_{(t_0+\Delta)-}\|_{W_2^{\gamma}((t_0+\Delta)-)(t_0,t_0+\Delta)} =& \Big(\int_{t_0}^{t_0+\Delta} \frac{|g(t_0+\Delta)-g(s)|^2}{|t_0+\Delta-s|^{2\gamma}}\,ds\Big)^{1/2} \\ &+ \Big(\int_{t_0}^{t_0+\Delta}\int_{t_0}^{t_0+\Delta} \frac{|g(r)-g(s)|^2}{|r-s|^{2\gamma+1}}\,ds\,dr\Big)^{1/2}\end{aligned}$$

First note that if g_{T-} is in $I^{\beta}_{T-}(L_2(0,T))$ then g is $(\beta-1/2)$ Hölder continuous on $(0,T)$ (see [31], Theorem 3.6). So, the first integral can be estimated as follows:

$$\begin{aligned}\Big(\int_{t_0}^{t_0+\Delta} \frac{|g(t_0+\Delta)-g(s)|^2}{|t_0+\Delta-s|^{2\gamma}}\,ds\Big)^{1/2} &\le C\Big(\int_{t_0}^{t_0+\Delta} \frac{|t_0+\Delta-s|^{2\beta-1}}{|t_0+\Delta-s|^{2\gamma}}\,ds\Big)^{1/2} \\ &\le C\Delta^{\beta-\gamma}.\end{aligned}$$

For the second integral we use the embedding $I^{\beta}_{T-}(L_2(0,T)) \hookrightarrow \widetilde{W}_2^{\gamma}(0,T)$ (see [8], Theorem 27). Therefore we can write with the notation $f(r,s) := \frac{|g(r)-g(s)|}{(r-s)^{\gamma+1/2}}$,

$$\begin{aligned}&\int_{t_0}^{t_0+\Delta}\int_{t_0}^{t_0+\Delta} \frac{|g(r)-g(s)|^2}{|r-s|^{2\gamma+1}}\,ds\,dr \\ &= \int_0^T\int_0^T \mathbf{1}\{(r,s)\in[t_0,t_0+\Delta]^2\} f(r,s)^2\,ds\,dr \\ &\le \int_0^T\int_0^T \mathbf{1}\{f^2 \ge N_\varepsilon\} f(r,s)^2\,ds\,dr \\ &\quad + \int_0^T\int_0^T \mathbf{1}\{(r,s)\in[t_0,t_0+\Delta]^2\}\mathbf{1}\{f^2 < N_\varepsilon\} f(r,s)^2\,ds\,dr,\end{aligned}$$

with a constant $N_\varepsilon \in (0,\infty)$. If we choose now N_ε large enough such that the first integral is less than $\frac{\varepsilon}{2}$ we get

$$\begin{aligned}&\int_0^T\int_0^T \mathbf{1}\{f^2 \ge N_\varepsilon\} f(r,s)^2\,ds\,dr \\ &+ \int_0^T\int_0^T \mathbf{1}\{(r,s)\in[t_0,t_0+\Delta]^2\}\mathbf{1}\{f^2 < N_\varepsilon\} f(r,s)^2\,ds\,dr\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\varepsilon}{2} + N_\varepsilon \int_{t_0}^{t_0+\Delta} \int_{t_0}^{t_0+\Delta} ds\, dr \\
&= \frac{\varepsilon}{2} + N_\varepsilon \Delta^2 < \varepsilon
\end{aligned}
$$

for $\Delta^2 < \frac{\varepsilon}{2N_\varepsilon}$. Hence, we proved that

$$
\int_{t_0}^{t_0+\Delta} \int_{t_0}^{t_0+\Delta} \frac{|g(r)-g(s)|^2}{|r-s|^{2\gamma+1}}\, ds\, dr \xrightarrow[\Delta\to 0]{} 0
$$

uniformly in t_0. □

Now we can formulate a *uniform local contraction principle* for the integral operator.

Theorem 3.5. *Let $x_0 \in \mathbb{R}^n, 1/2 < \beta' < \beta < 1, g_{T-} \in I^{\beta}_{T-}(L_2(0,T))$. Assume that the parameter function $\varphi \in W^{1/2-}_{2,\infty}(0,T)$ takes values in $\mathbb{R}^k$ and the mapping a fulfills hypothesis **(H1)** or **(H2)**. Then for any constant $c > 0$ there exists a $\Delta > 0$ such that for every interval $[t_0, t_1] \subset [0,T]$ with $t_1 - t_0 < \Delta$ the integral operator A given by*

$$
Af := x_0 + \int_{t_0}^{(\cdot)} a(f,\varphi)\, dg
$$

maps $W^{\beta'}_{2,\infty}(t_0,t_1)$ into itself and we have

$$
\|Af - Ah\|_{W^{\beta'}_{2,\infty}(t_0,t_1)} \leq c\|f-h\|_{W^{\beta'}_{2,\infty}(t_0,t_1)}
$$

for all vector functions $f, h \in W^{\beta'}_{2,\infty}(t_0,t_1)$.

Proof. Because of the Lipschitz property of the mapping a and Theorem 2.8 the integral operator A indeed maps the space $W^{\beta'}_{2,\infty}(t_0,t_1)$ into itself.
Theorem 3.3 implies for all $0 \leq t_0 < t \leq T$,

$$
\begin{aligned}
\|Af - Ah\|_{W^{\beta'}_{2,\infty}(t_0,t_1)} =& \left\| \int_{t_0}^{(\cdot)} a(f,\varphi)\, dg - \int_{t_0}^{(\cdot)} a(h,\varphi)\, dg \right\|_{W^{\beta'}_{2,\infty}(t_0,t)} \\
\leq& C\|f-h\|_{W^{\beta'}_{2,\infty}(t_0,t)} \|g_{t-}\|_{W^{\gamma}_2(t-)(t_0,t)},
\end{aligned}
$$

where $\frac{\beta+\beta'}{2} < \gamma < \beta$. From the last lemma we know that we can choose some small $\Delta > 0$ (independent of t_0) such that

$$C\|g_{t_1-}\|_{W_2^{\gamma}(t_1-)(t_0,t_1)} \leq c$$

for any $t_0 < t_1 \leq T$ with $t_1 - t_0 < \Delta$. $\square$

The statement of Theorem 3.5 remains valid if there is not only one integrator $g_{T-} \in I_{T-}^{\beta}(L_2(0,T))$.

Theorem 3.6. *Let $x_0 \in \mathbb{R}^n, 1/2 < \beta' < \beta < 1$. For $j = 1, ..., l$ the real-valued functions g_{T-}^j belong to the space $I_{T-}^{\beta}(L_2(0,T))$ and the mappings a_j fullfill hypothesis* **(*H*1)** *or* **(*H*1)**. *Further we assume that the parameter function $\varphi \in W_{2,\infty}^{1/2-}(0,T)$ takes values in $\mathbb{R}^k$. Then for any constant $c > 0$ there exists a $\Delta > 0$ such that for every interval $[t_0, t_1] \subset [0,T]$ with $t_1 - t_0 < \Delta$ the integral operator A given by*

$$Af := x_0 + \sum_{j=1}^{l} \int_{t_0}^{(\cdot)} a_j(f, \varphi)\, dg^j$$

maps $W_{2,\infty}^{\beta'}(t_0, t_1)$ into itself and we have

$$\|Af - Ah\|_{W_{2,\infty}^{\beta'}(t_0,t_1)} \leq c\|f - h\|_{W_{2,\infty}^{\beta'}(t_0,t_1)}$$

for all vector functions $f, h \in W_{2,\infty}^{\beta'}(t_0, t_1)$.

Proof. By the triangle inequality and Theorem 3.3 we get

$$\begin{aligned}
\|Af - Ah\|_{W_{2,\infty}^{\beta'}(t_0,t_1)} =& \left\| \sum_{j=1}^{l} \int_{t_0}^{(\cdot)} a_j(f,\varphi) - a_j(h,\varphi)\, dg^j \right\|_{W_{2,\infty}^{\beta'}(t_0,t)} \\
\leq& \sum_{j=1}^{l} \left\| \int_{t_0}^{(\cdot)} a_j(f,\varphi) - a_j(h,\varphi)\, dg^j \right\|_{W_{2,\infty}^{\beta'}(t_0,t)} \\
\leq& C \sum_{j=1}^{l} \|f - h\|_{W_{2,\infty}^{\beta'}(t_0,t)} \|g_{t-}^j\|_{W_2^{\gamma}(t-)(t_0,t)}.
\end{aligned}$$

If we choose now $\Delta > 0$ such that

$$C \sum_{j=1}^{l} \|g^j_{t_1-}\|_{W_2^\gamma(t_1-)(t_0,t_1)} \le c$$

for any $t_0 < t_1 \le T$ with $t_1 - t_0 < \Delta$ this concludes the proof. $\square$

Remark 3.7. In the case that $g^j(t) = t$ for some j the condition on $a_j(x,\tau)$ in the above theorem can be weakened to Lipschitz continuity in x and τ.

Theorem 3.6 provides a contraction principle for our differential equation (3.1) only on small intervalls. The next proposition is needed in order to extend the solution to the whole interval $[0,T]$.

Proposition 3.8. *Let $\Delta > 0$ be fixed, $0 < \beta < 1$ and $f : [0,T] \to \mathbb{R}^n$. If the restriction $f|_{[t_0,t_1]}$ of f belongs to $W^\beta_{2,\infty}(t_0,t_1)$ for any interval $[t_0,t_1] \subset [0,T]$ with $t_1 - t_0 < \Delta$, then the function f belongs to the space $W^\beta_{2,\infty}(0,T)$.*

Proof. Let $\{t_i\}_{i=0}^N$ be a partition of $[0,T]$ such that $t_0 = 0, t_N = T$ and $t_{i+1} - t_i = \widetilde{\Delta} < \Delta$ for $i = 0,...,N-1$, and let $\{\widetilde{t}_j\}_{j=0}^{N+1}$ be the partition of $[0,T]$ with $\widetilde{t}_0 = t_0 = 0, \widetilde{t}_{N+1} = t_N = T$ and $\widetilde{t}_j = \frac{t_{i+1}+t_i}{2}$ for $j = 1,...,N$. Every interval size of the second partition is also less than Δ. First note that

$$\|f\|_{L_\infty(0,T)} = \sup_{0 \le i \le N-1} \|f\|_{L_\infty(t_i,t_{i+1})} < \infty$$

and

$$\begin{aligned}
\|f\|^2_{\widetilde{W}_2^\beta(0,T)} &= \int_0^T \int_0^T \frac{|f(s)-f(r)|^2}{|s-r|^{2\beta+1}}\, dr\, ds \\
&= 2\int_0^T \int_0^s \frac{|f(s)-f(r)|^2}{(s-r)^{2\beta+1}}\, dr\, ds \\
&= 2\left(\int_0^T \int_0^{s-\varepsilon} \frac{|f(s)-f(r)|^2}{(s-r)^{2\beta+1}}\, dr\, ds + \int_0^T \int_{s-\varepsilon}^s \frac{|f(s)-f(r)|^2}{(s-r)^{2\beta+1}}\, dr\, ds\right).
\end{aligned}$$

The first summand can be bounded by

$$\begin{aligned}
\int_0^T \int_0^{s-\varepsilon} \frac{|f(s)-f(r)|^2}{(s-r)^{2\beta+1}}\, dr\, ds &\le C\|f\|^2_{L_\infty(0,T)} \int_0^T \int_0^{s-\varepsilon} (s-r)^{-2\beta-1}\, dr\, ds \\
&\le C\|f\|^2_{L_\infty(0,T)} T^2 \varepsilon^{-2\beta-1}.
\end{aligned}$$

If we take $\varepsilon = \widetilde{\Delta}/\sqrt{2}$, than we have for the second summand

$$
\begin{aligned}
&\int_0^T \int_{s-\varepsilon}^s \frac{|f(s)-f(r)|^2}{(s-r)^{2\beta+1}}\, dr\, ds \\
&\leq \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^s \frac{|f(s)-f(r)|^2}{(s-r)^{2\beta+1}}\, dr\, ds + \sum_{j=0}^{N} \int_{\widetilde{t}_j}^{\widetilde{t}_{j+1}} \int_{\widetilde{t}_j}^s \frac{|f(s)-f(r)|^2}{(s-r)^{2\beta+1}}\, dr\, ds \\
&= \sum_{i=0}^{N-1} \|f\|^2_{\widetilde{W}_2^\beta(t_i,t_{i+1})} + \sum_{j=0}^{N} \|f\|^2_{\widetilde{W}_2^\beta(\widetilde{t}_j,\widetilde{t}_{j+1})} \\
&< \infty,
\end{aligned}
$$

because of the assumption. □

3.3 Existence and uniqueness of a solution

Let us just recall the differential equation (3.1):

$$
\begin{aligned}
dx(t) &= \sum_{j=1}^m a_j(x(t), \varphi(t))\, dz^j(t) \\
x(0) &= x_0,
\end{aligned}
$$

which in integral representation means that

$$
x(t) = x_0 + \sum_{j=1}^m \int_0^t a_j(x(s), \varphi(s))\, dz^j(s)
$$

for $t \in [0, T], x_0 \in \mathbb{R}^n$ where the real-valued driving functions $z^1, \dots, z^m$ are such that $z^j_{T-} \in I^\beta_{T-}(L_2(0,T))$, $1/2 < \beta < 1$. The parameter function φ takes values in $\mathbb{R}^k$ with coordinate functions in $W^{1/2-}_{2,\infty}(0,T)$ and the a_j are $\mathbb{R}^n$-valued vector fields on $\mathbb{R}^n \times \mathbb{R}^k$ and satisfy the hypothesis ***(H1)*** or ***(H2)***, see Section 3.2. Applying the contraction theorem (Theorem 3.6) of the last section leads to the following result.

Theorem 3.9. *Under the above conditions there exists a global solution in* $W^{\beta-}_{2,\infty}(0,T)$ *of equation (3.1). It is unique in* $W^{\beta'}_{2,\infty}(0,T)$ *for any* $\beta' < \beta$.

Proof. According to Theorem 3.6 there exists a Δ such that for any interval $[r,s] \subset [0,T]$ with $s-r<\Delta$ the integral operator A given by (3.3) is a contraction in the space $W_{2,\infty}^{\beta'}(r,s)$ for any $1/2<\beta'<\beta$. Let $0=t_0<t_1<\ldots<t_N=T$ be a partition of the interval $[0,T]$ with $t_{i+1}-t_i<\Delta$. Setting $x(0)=x_0$ a global solution x to equation (3.1) can be determined successively by the contraction principle providing a unique solution $x(t), t\in[t_i,t_{i+1}]$, in the space $W_{2,\infty}^{\beta'}(t_i,t_{i+1})$ with initial condition $x(t_i)$. Let now $[r,s]\subset[0,T]$ with $s-r<\Delta$ and $t_{i-1}<r<t_i<s<t_{i+1}$ and y be the solution to (3.1) on $[r,s]$ with initial condition $y(r)=x_i(r)$. Then y and x are two solutions for (3.1) on $[r,t_i]$ with same initial condition and hence by the uniqueness from the contraction principle we have $y(t)=x_i(t)$ for all $t\in[r,t_i]$. With the same arguments we can show that $y(t)=x_{i+1}(t)$ for all $t\in[t_i,s]$ and therefore $y=1_{[r,s]}x$. So the solution of (3.1) is in $W_{2,\infty}^{\beta'}(r,s)$ for any interval $[r,s]\subset[0,T]$ with $s-r<\Delta$ and hence $x\in W_{2,\infty}^{\beta'}(0,T)$ by Proposition 3.8. The uniqueness of the global solution follows from the uniqueness for the local version. $\square$

Remark 3.10. If we choose $k=1, l=m+1, \varphi(t)=z_{m+1}(t)=t$ and $a_{m+1}=b$, then we obtain the differential equation in $\mathbb{R}^n$

$$\begin{aligned} dx(t) &= \sum_{j=1}^{m} a_j(x(t),t)\,dz^j(t)+b(x(t),t)\,dt \\ x(0) &= x_0. \end{aligned} \tag{3.4}$$

This special case of equation (3.1) will furhter be discussed in Chapter 5.

Chapter 4

Stochastic differential equations

This chapter is devoted to stochastic differential equations driven by fractal processes. In the first section we state the connection between the integral defined via fratal calculus from Chapter 2 and stochastic integrals. Afterwards we recall the definitions and properties about the generalized stochastic forward integral and the generalized covariation process, which were introduced by Zähle [39]. We end up with the well-known Itô formula and will give some modifications in special cases. Section 4.3 contains our main result. Combining a Doss-Sussman-type approach and the results from Chapter 3 we are able to solve pathwise stochastic differential equations. In the last section we give an example for processes, where the quadratic variation process in the sense of Zähle exist, whereas this is not true for the quadratic variation process in the sense of Russo and Vallois (see [27]).

4.1 Connection between stochastic integrals and fractional calculus

In this section we consider again real-valued functions on a finite interval $[a, b] \subset \mathbb{R}$. We start with a limit representation for the generalized Lebesgue-

Stieltjes integral defined in Section 2.4.

Lemma 4.1. *([39], Lemma 4.1) Let f and g be two real-valued functions such that $f(a+), g(a+)$ and $g(b-)$ exist, $f_{a+} \in I^{\alpha}_{a+}(L_p(a,b)), g_{b-} \in I^{1-\alpha}_{b-}(L_q(a,b))$ for some $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 \leq \alpha \leq 1$. Then the integral defined in (2.3) can be approximated as follows*

$$\int_a^b f\,dg = \lim_{\varepsilon \to 0} \int_a^b I^{\varepsilon}_{a+} f\,dg.$$

Here we used again the notation $f_{a+}(x) = \mathbf{1}_{(a,b)}(x)(f(x) - f(a+))$ and $f_{b-}(x) = \mathbf{1}_{(a,b)}(x)(f(x) - f(b-))$.
The integrals $\int_a^b f\,dg$ in the Lemma above are also determined if we assume for f and g a slightly lower order of differentiability. We denote

$$I^{\beta-}_{\substack{a+\\(b-)}}(L_p(a,b)) := \bigcap_{\alpha<\beta} I^{\alpha}_{\substack{a+\\(b-)}}(L_p(a,b)).$$

Lemma 4.2. *([39], Lemma 4.2) Let $\varepsilon > 0$ and $\frac{1}{p} + \frac{1}{q} \leq 1$. Then we get*

(i)
$$\int_a^b I^{\varepsilon}_{a+} f\,dg = \frac{1}{\Gamma(\varepsilon)} \int_0^{\infty} u^{\varepsilon-1} \int_a^b f(s) \frac{g_{b-}(s+u) - g_{b-}(s)}{u}\,ds\,du$$
provided that $f \in I^{\alpha-\varepsilon}_{a+}(L_p(a,b))$, $g_{b-} \in I^{1-\alpha}_{b-}(L_q(a,b))$ and $\alpha p \neq 1$.

(ii) The statement in (i) holds true if $f \in I^{\beta-}_{a+}(L_p(a,b))$ and $g_{b-} \in I^{(1-\beta)-}_{b-}(L_q(a,b))$ for some $0 < \beta < 1$.

The behaviour of the Gamma function allows to replace $1/\Gamma(\varepsilon)$ by ε if we consider the limit for ε tending to 0. This suggested the following extension of the integral (2.3) (see [39], Section 4):

Definition 4.3. For real-valued functions f and g we define the integral of f with respect to g by

$$\int_a^b f\,dg := \lim_{\varepsilon \to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_a^b f(s) \frac{g_{b-}(s+u) - g_{b-}(s)}{u}\,ds\,du \tag{4.1}$$

whenever the right-hand side exists.

Here the kernel $\varepsilon u^{\varepsilon-1}$ acts as a δ-function for $\varepsilon \to 0$. In the next section we replace the functions f and g in Definition 4.3 by stochastic processes and derive the so called generalized stochastic forward integral.

Remark 4.4. In the stochastic calculus in Section 4.2 the upper boundary b in the integrals may be replaced by any $t \in (a,b]$ and convergence holds uniformly in t.

4.2 Stochastic integrals, quadratic variation and Itô formula

For the rest of this chapter we alway consider a finite time interval $[0,T]$. The stochastic version of the Definition 4.3 reads as follows.

Definition 4.5. Let Y be a real-valued stochastic *càglàd process*, i.e. left continuous with right limits, and Z be a real-valued stochastic *càdlàg process* (right continuous with left limits) on $[0,T]$. The *generalized stochastic forward integral* of Y with respect to Z is defined by

$$\begin{aligned}\int_0^t Y\,dZ := \lim_{\substack{\varepsilon\to 0\\(ucp)}} \varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t Y(s) \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u}\,ds\,du \\ + Y(t)(Z(t) - Z(t-))\end{aligned} \tag{4.2}$$

whenever the right-hand side is determined, where $\lim_{(ucp)}$ stands for *uniform convergence in probability* and $\int_0^1$ for $\lim_{\delta\searrow 0}\int_\delta^1$ with probability 1. Notice that $Z_{t-}(s) = \mathbf{1}_{(0,t)}(s)(Z(s) - Z(t-))$.

We briefly recall the definition for ucp-convergence.

Definition 4.6. A family of processes $(H^\varepsilon(t))_{t\in[0,T]}$ is said to converge to a process $(H(t))_{t\in[0,T]}$ in the sense of uniform convergence in probability (ucp), if $\sup_{0\le t\le T} |H^\varepsilon(t) - H(t)|$ converges to 0 in probability, as $\varepsilon \to 0$.

The generalized stochastic forward integral as a stochastic process of the upper boundary

$$X(t) := \int_0^t Y \, dZ$$

is again a càdlàg process.

Russo and Vallois introduced in [25] and [26] a similar stochastic integral but without an averaging of the limit.

Definition 4.7. For a real-valued càglàd process Y and a real-valued stochastic càdlàg process Z the *stochastic forward integral* of Y with respect to Z in the sense of Russo and Vallois is defined by

$$\begin{aligned}(RV) - \int_0^t Y \, dZ := \lim_{\substack{u \to 0 \\ (ucp)}} \int_0^t Y(s) \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u} \, ds \\ + Y(t)(Z(t) - Z(t-))\end{aligned} \tag{4.3}$$

if the right-hand side exists.

The generalized stochastic forward integral is indeed an extension of the one introduced by Russo and Vallois by the following lemma.

Lemma 4.8. *Let Y and Z be two stochastic processes such that the stochastic forward integral $(RV) - \int_0^t Y \, dZ$ exists. Then also the generalized stochastic forward integral of Y with respect to Z exists and both integrals coincide, i.e.*

$$\int_0^t Y \, dZ = (RV) - \int_0^t Y \, dZ.$$

For a proof we refer to [41].

If the stochastic process Z is a semimartingale and Y is an adapted càglàd process then also the forward integral in the sense of Russo and Vallois (and hence the generalized stochastic forward integral) of Y with respect to Z exists and agrees with the usual Itô integral (see [27]).

Next we will define the generalized quadratic variation for a stochastic process.

Definition 4.9. A real-valued càdlàg process Z admits a *generalized quadratic variation process*

$$[Z](t) := \lim_{\substack{\varepsilon \to 0 \\ (ucp)}} \varepsilon \int_0^1 u^{\varepsilon - 1} \int_0^t \frac{(Z_{t-}(s+u) - Z_{t-}(s))^2}{u} \, ds \, du + (Z(t) - Z(t-))^2 \tag{4.4}$$

if the limit exists, where ucp stands again for uniform convergence in probability.

Note that the generalized quadratic variation process $[Z]$ is monotone and also càdlàg.

Russo and Vallois defined the *quadratic variation process* for a càdlàg process Z

$$(RV) - [Z](t) := \lim_{\substack{u \to 0 \\ (ucp)}} \int_0^t \frac{(Z_{t-}(s+u) - Z_{t-}(s))^2}{u} \, ds + (Z(t) - Z(t-))^2 \tag{4.5}$$

again without averaging the limit.

The existence of the quadratic variation of a càdlàg process Z implies again the existence of the generalized quadratic variation for Z. In the last section of this chapter we will show that the class of processes admitting a generalized quadratic variation is indeed greater then the one admitting a quadratic variation in the sense of Russo and Vallois.

For the special case when Z is a semimartingale these notations agree both with the classical one for Itô calculus, for example $[W](t) = t$, where W is a standrad Brownian motion. Fractional Brownian motion with Hurst exponent $1/2 < H < 1$ is another example for a process with generalized quadratic variation $[B^H] \equiv 0$. As a consequence of the definition of $[Z]$ is that any càdlàg process Z admitting a generalized quadratic variation belongs to the space $W^{1/2-}_{2,\infty}(0,T)$ with probability 1 (see [39], Prop. 5.3).

If the sample paths of a stochastic process are in some Hölder or fractional Sobolev space, then we derive also a generalized quadratic variation.

Proposition 4.10. *Let Z be a real-valued stochastic process such that $Z \in$*

$H^\beta(0,T)$ or $Z \in W_2^\beta(T-)(0,T)$ with probability 1 for some $\beta > 1/2$, then Z admits a generalized quadratic variation process and $[Z] \equiv 0$.

The *generalized covariation process* (or sometimes called *bracket*) for two real-valued càdlàg processes X and Z is defined by

$$\begin{aligned}
&[X,Z](t) \\
&:= \lim_{\substack{\varepsilon\to 0\\(ucp)}} \varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t \frac{(X_{t-}(s+u)-X_{t-}(s))(Z_{t-}(s+u)-Z_{t-}(s))}{u}\, ds\, du \\
&\qquad + (X(t)-X(t-))(Z(t)-Z(t-)).
\end{aligned} \tag{4.6}$$

Proposition 4.11. *([39], Prop. 5.4) Let X and Z be two real-valued continuous processes with generalized covariation $[X,Z]$ and we define $[X,Z]^\varepsilon$ by*

$$\begin{aligned}
[X,Z]^\varepsilon :=& \varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t \frac{(X_{t-}(s+u)-X_{t-}(s))(Z_{t-}(s+u)-Z_{t-}(s))}{u}\, ds\, du \\
&+ (X(t)-X(t-))(Z(t)-Z(t-)).
\end{aligned}$$

Then we have

$$\lim_{\substack{\varepsilon\to 0\\(ucp)}} \int_0^t Y\, d[X,Z]^\varepsilon = \int_0^t Y\, d[X,Z]$$

for any càglàd process Y on $[0,T]$.

Remark 4.12. Let X and Z be two càdlàg processes with generalized quadratic variation $[X]$ and $[Z]$ such that $[Z] \equiv 0$, then X and Z admit a generalized covariation $[X,Z]$ and $[X,Z] \equiv 0$.

For the multidimesional case we say that the stochastic processes $X = (X^1,...,X^n)$ and $Z = (Z^1,...,Z^m)$ with values in $\mathbb{R}^n$ and $\mathbb{R}^m$ admit a generalized covariation $[X,Z]$ if $[X^j,Z^k]$ exists for all $1 \le j \le n$ and $1 \le k \le m$.

Now we state that the generalized covariation is stable under C^1 transformations. See [39], Prop. 5.5 for the onedimesional case.

Proposition 4.13. *Let X and Z be two continuous processes with values in $\mathbb{R}^n$ and $\mathbb{R}^m$ admitting a generalized quadratic variation $[X],[Z]$ and the generalized covariation $[X,Z]$, and let $F:\mathbb{R}^n\to\mathbb{R}^p$ and $G:\mathbb{R}^m\to\mathbb{R}^q$ be two random C^1-functions. Then the processes $F(X)$ and $G(Z)$ admit again a generalized covariation given by*

$$[F^i(X),G^l(Z)](t)=\sum_{j=1}^n\sum_{k=1}^m\int_0^t\frac{\partial F^i}{\partial x^j}(X(s))\frac{\partial G^l}{\partial z^k}(Z(s))\,d[X^j,Z^k](s)$$

for all $1\le i\le p$ and $1\le l\le q$.

One of the most powerful tools for calculations in stochastic analysis is the so called *Itô formula*. It is the counterpart to the *change of variables formula* in ordinary calculus. As in the classical semimartingale theory we have the following Itô formula for the generalized stochastic forward integral.

Theorem 4.14. *([39], Thm. 5.10) Let $Z=(Z^1,...,Z^n)$ be a continuous $\mathbb{R}^n$-valued stochastic process admitting a generalized quadratic variation process $[Z]$, and let F be a random element of $C^1(\mathbb{R}^n\times[0,T],\mathbb{R}^p)$ with continuous partial derivatives $\frac{\partial^2 F}{\partial x^j\partial x^k}$, $1\le j,k\le n$. Then we have for any $t>0$*

$$\begin{aligned}F(Z(t),t)-F(Z(0),0)=&\sum_{j=1}^n\int_0^t\frac{\partial F}{\partial x^j}(Z(s),s)\,dZ^j(s)+\int_0^t\frac{\partial F}{\partial t}(Z(s),s)\,ds\\&+\frac{1}{2}\sum_{j,k=1}^n\int_0^t\frac{\partial^2 F}{\partial x^j\partial x^k}(Z(s),s)\,d[Z^j,Z^k](s).\end{aligned}\tag{4.7}$$

Remark 4.15. (i) We have to precise that we can not guarantee that each of the integrals $\int_s^t\frac{\partial F}{\partial x^j}(Z(s),s)\,dZ^j(s)$ exists, but the sum of the first n integrals is determined in the (ucp) sense.

(ii) The statement of Proposition 4.13 and Theorem 4.14 remains valid if the stochastic processes are càdlàg and not continuous. In that case there will appear an additional summand on the right hand sides of the formulas corresponding to the jumps (see Chapter 6).

As a direct consequence of Theorem 4.14 we infer the following result.

Corollary 4.16. *Let Z be a real-valued continuous process. The following properties are equivalent:*

(i) the generalized quadratic variation process $[Z]$ exists,

(ii) the generalized stochastic forward integral $\int_0^{(\cdot)} F(Z)\,dZ$ exists for all random functions $F \in C^1(\mathbb{R})$.

Definition 4.17. Let $Z = (Z^1, ..., Z^m)$ be a continuous $\mathbb{R}^m$-valued stochastic process admitting a generalized quadratic variation process $[Z]$, $Y_1, ..., Y_m$ are m càglàd processes with values in $\mathbb{R}^n$ and F as in Theorem 4.14. The $\mathbb{R}^n$-valued process X with

$$X(t) = \sum_{j=1}^{m} \int_0^t Y_j\,dZ^j$$

satisfies *the general Itô formula* if for any $t > 0$

$$\begin{aligned} &F(X(t),t) - F(X(0),0) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \int_0^t \frac{\partial F}{\partial x^i}(X(s),s) Y_j^i(s)\,dZ^j(s) + \int_0^t \frac{\partial F}{\partial t}(X(s),s)\,ds \\ &\quad + \frac{1}{2}\sum_{i,l=1}^{n}\sum_{j,k=1}^{m} \int_0^t \frac{\partial^2 F}{\partial x^i \partial x^l}(X(s),s) Y_j^i(s) Y_k^l(s)\,d[Z^j, Z^k](s). \end{aligned} \tag{4.8}$$

The condition in the Itô formula on the random function F to have continuous partial derivatives of order 2 can be weakened in the case when the generalized quadratic variation processes are equal to zero.

Lemma 4.18. *Let $Z = (Z^1, ..., Z^n)$ be a continuous $\mathbb{R}^n$-valued process admitting a generalized quadratic variation process $[Z]$ such that $[Z^j, Z^k] \equiv 0$ for all $1 \le j,k \le n$, and let F be a random element of $C^1(\mathbb{R}^n, \mathbb{R}^p)$ such that the partial derivatives $\frac{\partial F}{\partial x^j}$ are Lipschitz continuous for $1 \le j \le n$. Then we have for any $t > 0$*

$$F(Z(t)) - F(Z(0)) = \sum_{j=1}^{n} \int_0^t \frac{\partial F}{\partial x^j}(Z(s))\,dZ^j(s). \tag{4.9}$$

Proof. For simplicity we just prove the case $p = n = 1$. As in the proof for the Itô formula (4.7) we use the Taylor expension for the function F,

$$F(Z(s+u)) - F(Z(s)) = \frac{\partial F}{\partial x}(Z(s))(Z(s+u) - Z(s)) + R(s,u).$$

By the mean value theorem we can write for the remainder term

$$R(s,u) = \left(\int_0^1 \frac{\partial F}{\partial x}(\alpha Z(s+u) + (1-\alpha)Z(s)) - \frac{\partial F}{\partial x}(Z(s))\, d\alpha\right)(Z(s+u) - Z(s))$$

and get the estimate

$$\begin{aligned}
&|R(s,u)| \\
&\leq \left(\int_0^1 \left|\frac{\partial F}{\partial x}(\alpha Z(s) + (1-\alpha)Z(s+u)) - \frac{\partial F}{\partial x}(Z(s))\right| d\alpha\right)|Z(s+u) - Z(s)| \\
&\leq C\left(\int_0^1 \left|(1-\alpha)(Z(s+u) - Z(s))\right| d\alpha\right)|Z(s+u) - Z(s)| \\
&\leq C|Z(s+u) - Z(s)|^2
\end{aligned}$$

and hence

$$\lim_{\substack{\varepsilon \to 0 \\ (ucp)}} \varepsilon \int_0^1 u^{\varepsilon - 1} \int_0^t \frac{1}{u} R(s,u)\, ds\, du = 0.$$

The continuity of the process Z implies

$$\lim_{\substack{\varepsilon \to 0 \\ (ucp)}} \varepsilon \int_0^1 u^{\varepsilon - 1} \int_0^t \frac{F(Z(s+u)) - F(Z(s))}{u}\, ds\, du = F(Z(t)) - F(Z(0)).$$

So the ucp limit of

$$\begin{aligned}
&\varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t \frac{\partial F}{\partial x}(Z(s)) \frac{(Z(s+u) - Z(s))}{u}\, ds\, du \\
&= \varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t \frac{\partial F}{\partial x}(Z(s)) \frac{(Z_{t-}(s+u) - Z_{t-}(s))}{u}\, ds\, du
\end{aligned}$$

for $\varepsilon \to 0$ exists and is by defintion the generalized stochastic forward integral

$$\int_0^t \frac{\partial F}{\partial x}(Z(s))\, dZ(s).$$

The proof for the higherdimensional case is straightforward. $\square$

In the special case for stochastic processes with a.s. Hölder continuous paths of order greater 1/2 we have the following result.

Lemma 4.19. *Let $Z = (Z^1, ..., Z^n)$ be a $\mathbb{R}^n$-valued process with Hölder continuous paths of order $\beta > 1/2$ and let F be a random element of $C^1(\mathbb{R}^n, \mathbb{R}^p)$ such that the partial derivatives $\frac{\partial F}{\partial x^j}$ are Hölder continuous of order μ with $(\mu+1)\beta > 1$ for $1 \leq j \leq n$. Then we have for any $t > 0$*

$$F(Z(t)) - F(Z(0)) = \sum_{j=1}^{n} \int_0^t \frac{\partial F}{\partial x^j}(Z(s))\, dZ^j(s). \tag{4.10}$$

Proof. For the proof we refer to [20], Theorem 2.7.3. □

We will later need the following special version of Itô's formula. Suppose that $F(y,z) : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is a random C^1-function with continuous $\frac{\partial^2 F}{\partial z^2}$ and $\frac{\partial F}{\partial y}$ is Lipschitz in y. The $\mathbb{R}^m$-valued random process $Y = (Y^1, ..., Y^m)$ has sample paths such that $Y^i \in W_{2,\infty}^{\beta'}(0,T)$ for some $\beta' > 1/2$ and Z is a real-valued stochastic process with generalized quadratic variation $[Z]$. Hence it follows that $[Y^j, Y^k] \equiv 0$ for $1 \leq j, k \leq m$ and also $[Y^i, Z] \equiv 0$ for $i = 1, ..., m$. Then we obtain for any $t > 0$ (see [42])

$$\begin{aligned} F(Y(t), Z(t)) - F(Y(0), Z(0)) = &\sum_{i=1}^{m} \int_0^t \frac{\partial F}{\partial y^i}(Y(s), Z(s))\, dY^i(s) \\ &+ \int_0^t \frac{\partial F}{\partial z}(Y(s), Z(s))\, dZ(s) \\ &+ \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial z^2}(Y(s), Z(s))\, d[Z](s). \end{aligned} \tag{4.11}$$

Here the first m integrals are determined in the sense of (2.3) and the integral with respect to Z is given by the generalized stochastic forward integral (4.2). Note that we may choose, in particular, $Y^m(t) = t$ in order to apply the formula to SDE's with time dependent coefficients.

4.3 Main result

Througout this section let $Z^0, Z^1, \dots, Z^m$ be real-valued continuous processes on $[0,T]$ such that Z^0 admits a generalized quadratic variation process $[Z^0]$ and the processes $[Z^0]_T, Z_T^1, ..., Z_T^m$ have sample paths in $I_{T-}^{\beta}(L_2(0,T))$ for some $1/2 < \beta < 1$. (Here we denote $Z_T(t) := \mathbf{1}_{[0,T]}(t)(Z(t) - Z(T))$. We consider the stochastic differential equation in $\mathbb{R}^n$,

$$\begin{aligned} dX(t) &= \sum_{j=0}^{m} a_j(X(t),t)\, dZ^j(t) + b(X(t),t)\, dt \\ X(0) &= X_0 \end{aligned} \tag{4.12}$$

for certain random vector fields $a_0, a_1, \dots a_m, b : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ and an arbitrary random initial vector $X_0 \in \mathbb{R}^n$.

Definition 4.20. A solution of (4.12) is a continuous process $X = (X^1, \dots, X^n)$ admitting a generalized quadratic variation processes $[X]$, which satisfies the multidimensional generalized version of the Itô formula (4.8) with respect to its coordinatewise integral representation

$$X(t) = X_0 + \sum_{j=0}^{m} \int_0^t a_j(X(s),s)\, dZ^j(s) + \int_0^t b(X(s),s)\, ds,$$

where the first $m+1$ integrals are defined in the sense of (4.2).

For the vector fields a_j and b we assume (with probability 1)

(C1) $a_j : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ and satisfy hypothesis **(*H*1)** or **(*H*2)** from Section 3.2, $j = 1, ..., m$,

(C2) $a_0 \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$, the partial derivatives $\frac{\partial a_0}{\partial x}(x,t)$ and $\frac{\partial a_0}{\partial t}(x,t)$ are Lipschitz in x, and $\frac{\partial a_0}{\partial x} a_0$ also satisfy hypothesis **(*H*1)** or **(*H*2)** [1],

(C3) $b \in C(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$, $b(x,t)$ is Lipschitz continuous in x and Hölder continuous of order γ in t for some $\gamma > \beta$.

[1] This condition is missing in [42]

To determine a pathwise global solution we consider the vector-valued auxiliary ordinary differential equation on $\mathbb{R}^n \times K \times [0,T]$

$$\begin{aligned} \frac{\partial h}{\partial z}(y,z,t) &= a_0(h(y,z,t),t) \\ h(y,Z_0,t) &= y, \end{aligned} \tag{4.13}$$

where K is a compact subset of $\mathbb{R}$ containing $Z_0 = Z^0(0)$. If a_0 is a random function then the above equation holds true almost surely. The classical theory of ODE (see e.g. [36]) provides the existence of a C^1-solution which satisfies

$$h(y,z,t) = y + \int_{Z_0}^{z} a_0(h(y,u,t),t)\,du.$$

For the other partial derivatives of h we get in matrix representation

$$\begin{aligned} \frac{\partial h}{\partial y}(y,z,t) =& \exp\left\{\int_{Z_0}^{z} \frac{\partial a_0}{\partial x}(h(y,u,t),t)\,du\right\}, \\ \frac{\partial h}{\partial t}(y,z,t) =& \exp\left\{\int_{Z_0}^{z} \frac{\partial a_0}{\partial x}(h(y,u,t),t)\,du\right\} \\ & \left(\int_{Z_0}^{z} \frac{\partial a_0}{\partial t}(h(y,u,t),t)\exp\left\{-\int_{Z_0}^{u} \frac{\partial a_0}{\partial x}(h(y,v,t),t)\,dv\right\}du\right). \end{aligned}$$

In particular, the mapping $h(\cdot,z,t) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible with inverse function $u(x,z,t)$, i.e.

$$u(h(y,z,t),z,t) = y.$$

Moreover,

$$\frac{\partial^2 h}{\partial z^2}(y,z,t) = \frac{\partial a_0}{\partial x}(h(y,z,t),t)a_0(h(y,z,t),t).$$

We will seek the solution X of (4.12) in the form

$$X(t) = h(Y(t),Z^0(t),t)$$

for some random $W_{2,\infty}^{\beta-}$-process Y with $Y(0) = X_0$.
Applying the Itô formula (4.11) to the function h we obtain

$$\begin{aligned} dX(t) = \quad & \frac{\partial h}{\partial z}(Y(t),Z^0(t),t)dZ^0(t) + \sum_{k=1}^{n} \frac{\partial h}{\partial y^k}(Y(t),Z^0(t),t)dY^k(t) \\ & + \frac{\partial h}{\partial t}(Y(t),Z^0(t),t)dt + \frac{1}{2}\frac{\partial^2 h}{\partial z^2}(Y(t),Z^0(t),t)d[Z^0](t) \end{aligned}$$

$$
\begin{aligned}
=&a_0(X(t),t)dZ^0(t) + \sum_{k=1}^{n} \frac{\partial h}{\partial y^k}(Y(t),Z^0(t),t)dY^k(t) + \frac{\partial h}{\partial t}(Y(t),Z^0(t),t)dt \\
&+ \frac{1}{2}\sum_{k=1}^{n} \frac{\partial a_0}{\partial x^k}(h(Y(t),Z^0(t),t),t)a_0^k(h(Y(t),Z^0(t),t),t)d[Z^0](t).
\end{aligned}
$$

Comparing this with (4.12) we are led to a second auxiliary SDE, now for the process Y (in matrix representation):

$$
\begin{aligned}
dY(t) = &\left(\frac{\partial h}{\partial y}(Y(t),Z^0(t),t)\right)^{-1} \Bigg[\sum_{j=1}^{m} a_j(h(Y(t),Z^0(t),t),t)\, dZ^j(t) \\
&+ \Big(b(h(Y(t),Z^0(t),t),t) - \frac{\partial h}{\partial t}(Y(t),Z^0(t),t)\Big)\, dt \\
&- \frac{1}{2}\frac{\partial a_0}{\partial x}(h(Y(t),Z^0(t),t),t)a_0(h(Y(t),Z^0(t),t),t)\, d[Z^0](t)\Bigg].
\end{aligned}
\tag{4.14}
$$

Using Theorem 3.9 we know that this differential equation has a pathwise unique solution $Y \in W_{2,\infty}^{\beta-}(0,T)$ which may be determined via Picard's iteration method.

Theorem 4.21. *Under the conditions* ***(C1)*** *-* ***(C3)*** *for the random vector fields the process*

$$X(t) = h(Y(t),Z^0(t),t)$$

with h and Y satisfying (4.13) and (4.14) provides a pathwise global solution of the SDE (4.12).
If X is an arbitrary solution of the SDE (4.12) in the sense of Definition 4.20 then it agrees with the above representations on the whole interval $[0,T]$.

Proof. By construction the process $h(Y(t),Z^0(t),t)$ is indeed a solution to the SDE (4.12). For uniqueness we take another solution $X(t)$ and $h(Y(t),Z^0(t),t)$ from above. As mentioned before the mapping

$$(y,z,t) \mapsto (h(y,z,t),z,t)$$

is invertible (a.s.). If we denote the inverse mapping by $(u(x,z,t),z,t)$, i.e.

$$u(h(y,z,t),z,t) = y,$$

then the matrix equality

$$\left(\frac{\partial u}{\partial x}(x,z,t)\right) = \left(\frac{\partial h}{\partial y}(u(x,z,t),z,t)\right)^{-1}$$

holds true and furthermore,

$$\begin{aligned}
\frac{\partial u}{\partial z}(x,z,t) &= -\sum_{k=1}^{n} \frac{\partial u}{\partial x^k}(x,z,t) a_0^k(x,t) \\
\frac{\partial u}{\partial t}(x,z,t) &= -\sum_{k=1}^{n} \frac{\partial u}{\partial x^k}(x,z,t) \frac{\partial h^k}{\partial t}(u(x,z,t),z,t) \\
\frac{\partial^2 u}{\partial z^2}(x,z,t) &= -\sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial x^k \partial x^l}(x,z,t) a_0^k(x,t) a_0^l(x,t) \\
&\quad - \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z \partial x^k}(x,z,t) a_0^k(x,t) - \sum_{k,l=1}^{n} \frac{\partial u}{\partial x^k}(x,z,t) \frac{\partial a^k}{\partial x^l}(x,t) a_0^l(x,t).
\end{aligned}$$

Applying the Itô formula to the function $u(x,z,t)$ and the process $(X(t), Z^0(t))$ given by the integral representation

$$X(t) = X_0 + \sum_{j=0}^{m} \int_0^t a_j(X(s),s)\, dZ^j(s) + \int_0^t b(X(s),s)\, ds,$$

leads to

$$\begin{aligned}
&u(X(t), Z^0(t), t) - X_0 \\
&= \int_0^t \frac{\partial u}{\partial z}(X(s), Z^0(s), s)\, dZ^0(s) + \int_0^t \frac{\partial u}{\partial t}(X(s), Z^0(s), s)\, ds \\
&\quad + \sum_{k=1}^{n} \int_0^t \frac{\partial u}{\partial x^k}(X(s), Z^0(s), s) b^k(X(s), s)\, ds \\
&\quad + \sum_{j=0}^{m} \sum_{k=1}^{n} \int_0^t \frac{\partial u}{\partial x^k}(X(s), Z^0(s), s) a_j^k(X(s), s)\, dZ^{j,k}(s) \\
&\quad + \frac{1}{2} \sum_{l,k=1}^{n} \int_0^t \frac{\partial^2 u}{\partial x^l \partial x^k}(X(s), Z^0(s), s) a_0^l(X(s), s) a_0^k(X(s), s)\, d[Z^0](s) \\
&\quad + \frac{1}{2} \int_0^t \frac{\partial^2 u}{\partial z^2}(X(s), Z^0(s), s)\, d[Z^0](s) \\
&\quad + \frac{1}{2} \sum_{k=1}^{n} \int_0^t \frac{\partial^2 u}{\partial x^k \partial z}(X(s), Z^0(s), s) a_0^k(X(s), s)\, d[Z^0](s).
\end{aligned}$$

Substituting the expressions from above we infer

$$
\begin{aligned}
&u(X(t), Z^0(t), t) - X_0 \\
=&\sum_{j=1}^{m}\sum_{k=1}^{n}\int_0^t \frac{\partial u}{\partial x^k}(X(s), Z^0(s), s) a_j^k(X(s), s)\, dZ^{j,k}(s) \\
&+ \sum_{k=1}^{n}\int_0^t \frac{\partial u}{\partial x^k}(X(s), Z^0(s), s) b^k(X(s), s)\, ds \\
&- \sum_{k=1}^{n}\int_0^t \frac{\partial u}{\partial x^k}(X(s), Z^0(s), s)\frac{\partial h^k}{\partial t}(u(X(s), Z^0(s), s), Z^0(s), s)\, ds \\
&- \frac{1}{2}\sum_{l,k=1}^{n}\int_0^t \frac{\partial u}{\partial x^k}(X(s), Z^0(s), s)\frac{\partial a_0^k}{\partial x^l}(X(s), s) a_0^l(X(s), s)\, d[Z^0](s).
\end{aligned}
$$

Together with the relation $\frac{\partial h}{\partial z}(y, z, t) = a_0(h(y, z, t), z, t)$ it follows that the process

$$\widetilde{Y}(t) := u(X(t), Z^0(t), t)$$

has sample paths in $W_{2,\infty}^{1/2-}(0, T)$ and satisfies the SDE (4.14):

$$
\begin{aligned}
d\widetilde{Y}(t) =& \left(\frac{\partial h}{\partial y}(\widetilde{Y}(t), Z^0(t), t)\right)^{-1} \Bigg[\sum_{j=1}^{m} a_j(h(\widetilde{Y}(t), Z^0(t), t), t)\, dZ^j(t) \\
&+ \left(b(h(\widetilde{Y}(t), Z^0(t), t), t) - \frac{\partial h}{\partial t}(\widetilde{Y}(t), Z^0(t), t)\right) dt \\
&- \frac{1}{2}\frac{\partial a_0}{\partial x}(h(\widetilde{Y}(t), Z^0(t), t), t) a_0(h(\widetilde{Y}(t), Z^0(t), t), t)\, d[Z^0](t)\Bigg] \\
\widetilde{Y}(0) =& X_0.
\end{aligned}
$$

Since the solution of (4.14) is unique in $W_{2,\infty}^{1/2-}(0, T)$ we get

$$\widetilde{Y}(t) = u(X(t), Z^0(t), t) = Y(t)$$

and hence

$$X(t) = h(Y(t), Z^0(t), t).$$

This completes the proof. □

Remark 4.22. (i) Applying Proposition 4.13 we can express the quadratic variation process $[X]$ for the solution of the SDE (4.12) by

$$[X^i, X^j](t) = \int_0^t a_0^i(X(s), s) a_0^j(X(s), s)\, d[Z^0](s)$$

for all $1 \leq i, j \leq n$.

(ii) If the generalized quadratic variation process of Z^0 has the form $[Z^0](t) = ct$ for some constant $c \in (0, \infty)$, then the condition **(C2)** can be relaxed to $a_0(x,t)$ is differentiable in x and $\frac{\partial a_0}{\partial x} a_0(x,t)$ is Lipschitz continuous in x and Hölder continuous of order γ in t for some $\gamma > \beta$.

(iii) For $n = 1, m = 0$ and $\det(\frac{\partial a_0}{\partial x}(X_0, t_0)) \neq 0$ the special version

$$h(y, z, t) = \widetilde{h}(y + z, t)$$

with

$$\begin{aligned} \frac{\partial \widetilde{h}}{\partial z}(z,t) &= a_0(\widetilde{h}(z,t),t) \\ \widetilde{h}(Z_0, t_0) &= X_0 \end{aligned}$$

has been treated in [13].

4.4 Random Weierstrass-type functions

In this section we want to present a class of random processes such that the generalized quadratic variation process defined by (4.4) exists, whereas the quadratic variation process in the sense of Russo and Vallois (see (4.5)) does not.

The *Weierstrass function* is given by

$$W(x) := \sum_{k=1}^{\infty} 2^{-\frac{k}{2}} \sin(2^k x), \quad x \in \mathbb{R} \tag{4.15}$$

This function was first analysed in 1872 by Karl Weierstraß as an example for a continuous function that is nowhere differentiable. It is also known that the Weisestrass function is Hölder continuous of order 1/2 (see [7]) and has period 2π. Moreover this function is representative of a much wider class of functions, the so called *self-affine functions*. For a rigorous definition of self-affine functions we refer to [7].
Note that the Weierstrass function is self-affine of order 1/2.

Theorem 4.23. *([2], Thm. 5) Let $f : \mathbb{R} \to \mathbb{R}$ a self-affine function of order α, then there exists a constant $\lambda > 0$ (depending on f) such that*

$$\lim_{T\to\infty} \int_0^T \frac{|f(x+e^{-t}) - f(x)|^{1/\alpha}}{e^{-t}}\, dt = \lambda \tag{4.16}$$

for a.a. $x \in \mathbb{R}$.

Remark 4.24. Substituting $e^{-t} = u$ the left hand side in the last theorem can be rewritten as

$$\lim_{T\to\infty} \int_0^T \frac{|f(x+e^{-t}) - f(x)|^{1/\alpha}}{e^{-t}}\, dt = \lim_{\delta\to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \frac{(f(x+u) - f(x))^{1/\alpha}}{u} \frac{1}{u}\, du.$$

Let $(\xi_k)_{k\geq 1}$ be a stationary sequence of random variables with second moments such that $\sup_k |\xi_k| < \infty$ w.p. 1. Then we define the *random Weierstrass-type function*

$$W(s) := \sum_{k=1}^{\infty} \xi_k 2^{-\frac{k}{2}} \sin(2^k s) \tag{4.17}$$

on $\mathbb{R}$ which is also Hölder continuous of order $1/2$ and has again period 2π.

Proposition 4.25. *There exists a constant $K > 0$ such that w.p. 1,*

$$\lim_{\delta\to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \frac{(W(s+u) - W(s))^2}{u} \frac{1}{u}\, du = K$$

for Lebesgue almost all $s \in [2, \pi)$.

Remark 4.26. Since the averaging kernel $\varepsilon u^{\varepsilon-1}$ arises from the last kernel $\frac{1}{|\ln\delta|}\mathbf{1}_{(\delta,1)}(u)\frac{1}{u}$ by means of a second averaging kernel $\varepsilon^2 v^{\varepsilon-1}|\ln v|$ (see [41], Prop. 3.1) this implies

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \frac{(W(s+u) - W(s))^2}{u}\, du = K \tag{4.18}$$

for almost all s w.p. 1.

Corollary 4.27. *The generalized quadratic variation process $[W]$ for the random Weierstrass-type function W exists and $[W](t) = Kt$, i.e.*

$$\lim_{\varepsilon\to 0}\left(\varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t \frac{(W_{t-}(s+u) - W_{t-}(s))^2}{u}\, ds\, du - Kt \right) = 0$$

uniformly in t w.p. 1.

Below we outline that the limit of

$$\mathbb{E}\int_0^{2\pi} \frac{(W(s+2\varepsilon)-W(s))^2}{2\varepsilon}\,ds$$

as $\varepsilon \to 0$ does not exist. (Here we write 2ε instead of ε to get simpler expressions.) So,

$$I(\varepsilon) := \int_0^{2\pi} \frac{(W(s+2\varepsilon)-W(s))^2}{2\varepsilon}\,ds$$

cannot converge in probability, provided the random variables $I(\varepsilon)$ are integrable uniformly in ε. The latter is guaranteed if the $(\xi_k)_{k\geq 1}$ are uniformly square integrable. Therefore in this case we conclude that the process $W(t)$ has no quadratic variation in the sense of Russo and Vallois and does not fit into the approach of [27] and [29].

By the trigonometric identity

$$\sin(x)-\sin(y) = 2\cos\Big(\frac{x+y}{2}\Big)\sin\Big(\frac{x-y}{2}\Big)$$

we infer

$$2\varepsilon I(\varepsilon) = \int_0^{2\pi}\Bigg(\sum_{k=1}^{\infty} 2\xi_k 2^{-\frac{k}{2}}\cos(2^k(s+\varepsilon))\sin(2^k\varepsilon)\Bigg)^2\,ds.$$

Note that for $k, n \in \mathbb{N}$ with $n \neq k$

$$\int_0^{2\pi}\cos(kx)\cos(nx)\,dx = 0.$$

So we get

$$I(\varepsilon) = \frac{1}{\varepsilon}\sum_{k=1}^{\infty} 2\xi_k^2 2^{-k}\sin^2(2^k\varepsilon)\int_0^{2\pi}\cos^2(2^k(s+\varepsilon))\,ds$$

because of the orthogonality of the involved trigonometric functions. The integral can be calculated by using that

$$\int_0^{2\pi}\cos^2(2^k(s+\varepsilon))\,ds = \pi.$$

If we define $m := \mathbb{E}\xi_k^2$, then

$$\varepsilon \mathbb{E} I(\varepsilon) = m \sum_{k=1}^{\infty} 2\pi 2^{-k} \sin^2(2^k \varepsilon).$$

This is again a Weierstrass-type function. Therefore the limit of

$$\mathbb{E} I(\varepsilon) = C \frac{1}{\varepsilon} \sum_{k=1}^{\infty} 2^{-k} \sin^2(2^k \varepsilon)$$

as $\varepsilon \to 0$ cannot exist.

Chapter 5

Hölder continuity of the solution

In Chapter 3 we proved the existence and uniqueness for the differential equation (3.1) in $\mathbb{R}^n$. Now we analyse the Hölder regularity of the solution when the integrator functions $g^1, ..., g^m$ are Hölder continuous of order $\beta > 1/2$. In the first section we recall a result from the literature and in the second section we present a more general result about the optimal degree of Hölder regularity. This will also be applied to the stochastic differential equation (4.12) from Chapter 4.

5.1 The result of Nualart and Răşcanu

In [21] the following deterministic differential equation on $\mathbb{R}^n$ was considered:

$$x(t) = x_0 + \sum_{j=1}^{m} \int_0^t \sigma_j(x(s), s)\, dg^j(s) + \int_0^t b(x(s), s)\, ds, \tag{5.1}$$

where the vector function b fulfills

$$\begin{aligned} |b(x,t) - b(y,t)| &\leq C|x-y|, \\ |b(x,t)| &\leq C(1+|x|) \end{aligned} \tag{5.2}$$

and the vector-valued functions σ_j satisfy

$$\begin{aligned} |\sigma_j(x,t) - \sigma_j(y,t)| &\leq C|x-y|, \\ \Big|\frac{\partial \sigma_j}{\partial x^i}(x,t) - \frac{\partial \sigma_j}{\partial y^i}(y,t)\Big| &\leq C|x-y|^{\delta_1}, \\ |\sigma_j(x,t) - \sigma_j(x,s)| &\leq C|t-s|^{\delta_2}, \\ \Big|\frac{\partial \sigma_j}{\partial x^i}(x,t) - \frac{\partial \sigma_j}{\partial x^i}(x,s)\Big| &\leq C|t-s|^{\delta_3}, \end{aligned} \tag{5.3}$$

for all $x, y \in \mathbb{R}^n$, $i = 1, ..., n$, $t, s \in [0, T]$, $j = 1, ..., m$, and some constants $0 < \delta_1, \delta_2, \delta_3 \leq 1$, C. Here the integrals are understood in the same sense as in (2.3). This is a special case of our differential equation (3.1) from Chapter 3 (see Remark 3.10), but here the vector-valued functions $\sigma_1, ..., \sigma_m$ and b are slightly different. For $0 < \alpha < 1/2$ those authors considered the space $W_T^{1-\alpha,\infty}(0,T)$ of vector functions $f : [0,T] \to \mathbb{R}^n$ such that

$$\|f\|_{1-\alpha,\infty,T} := \sup_{0<s<t<T} \left(\frac{|f(t) - f(s)|}{(t-s)^{(1-\alpha)}} + \int_s^t \frac{|f(y) - f(s)|}{(y-s)^{2-\alpha}}\, dy \right) < \infty$$

and the Banach space $W_0^{\alpha,\infty}$ with respect to the norm

$$\|f\|_{\alpha,\infty} := \sup_{t\in[0,T]} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}}\, ds \right).$$

For the space $W_T^{1-\alpha,\infty}(0,T)$ we have the embeddings

$$H^{1-\alpha+\varepsilon}(0,T) \hookrightarrow W_T^{1-\alpha,\infty}(0,T) \hookrightarrow H^{1-\alpha}(0,T),$$

for every $\varepsilon > 0$. Nualart and Răşcanu proved the following.

Theorem 5.1. *([21], Thm. 2.1) Let $g^j \in W_T^{1-\alpha,\infty}(0,T)$, the functions b and σ_j satisfy the conditions (5.3) for $j = 1, ..., m$, and $0 < \alpha < \alpha_0 = \min\left(\frac{1}{2}, \delta_2, \delta_3, \frac{\delta_1}{1+\delta_1}\right)$. Then Equation (5.1) has a unique solution $x \in W_0^{\alpha,\infty}$. Moreover, the solution is $(1-\alpha)$-Hölder continuous.*

So for Hölder continuous functions g^j of order $\beta > 1/2$ the above theorem only implies (if $1 - \beta < \min(1/2, \delta_2, \delta_3, \frac{\delta_1}{\delta_1+1})$) that (5.1) has a β-Hölder continuous solution x if the functions g^j are in the smaller space $W_T^{\beta,\infty}(0,T)$, and we only get a $(\beta - \varepsilon)$-Hölder continuous solution in the other case.

5.2 The optimal degree of Hölder continuity

The aim of this section is to extend the above result. We will show that the solution of (5.1) is Hölder continuous of the same order as the integrator functions $g^j \in H^\beta(0,T)$. In Chapter 3 we already considered a similar equation (see Equation (3.4)), but under the hypotheses $(\boldsymbol{H1})$ and $(\boldsymbol{H2})$ on the coefficients $\sigma_1, ..., \sigma_m$. First, we prove the optimal degree of Hölder regularity for (5.1) with the conditions from Chapter 3 and afterwards we conclude the same result for Equation (5.1) under the Conditions (5.3). The basic tool to this aim are the following relationships.

Proposition 5.2. *Let g be a real-valued β-Hölder continuous function with $1/2 < \beta \leq 1$ and $f \in W_2^\gamma(0,T)$ for some $\gamma > 1-\beta$ (and hence $f_{0+} \in I_{0+}^{\gamma-}(L_2(0,T))$). The function h is defined by*

$$h(t) := \int_0^t f(s)\, dg(s) \tag{5.4}$$

and the integral is determined in the sense of (2.3). Then we have the following:

(i) The function h is η-Hölder continuous for all $\eta < \beta - 1/2$.

(ii) If $\gamma > 1/2$ and f is δ-Hölder continuous for some $0 < \delta \leq 1$, then h is β-Hölder continuous.

(iii) If $\gamma < 1/2$ and f is δ-Hölder continuous for some $0 < \delta \leq 1$, then h is Hölder continuous of all orders $\mu < \beta + \gamma - 1/2$.

Proof. (i) Theorem 2.5 in [38] implies

$$\begin{aligned}|h(t) - h(s)| &= \Big| \int_s^t D_{s+}^\alpha f(x) D_{t-}^{1-\alpha} g_{t-}(x)\, dx \Big| \\ &\leq \sup_{0 \leq s < x < t \leq T} |D_{t-}^{1-\alpha} g_{t-}(x)| \int_s^t |D_{s+}^\alpha f(x)|\, dx \\ &\leq \|g_{t-}\|_{I_{T-}^{1-\alpha}(L_\infty(0,T))} \int_s^t |D_{s+}^\alpha f(x)|\, dx\end{aligned}$$

for all $1-\beta<\alpha<\gamma$.
The integral does not exceed

$$\begin{aligned}\int_s^t |D^\alpha_{s+}f(x)|\,dx &\le C\int_s^t \frac{|f(x)|}{(x-s)^\alpha}\,dx+\int_s^t\int_s^x \frac{|f(x)-f(u)|}{(x-u)^{\alpha+1}}\,du\,dx\\ &=: S_1+S_2.\end{aligned}$$

Using the Hölder inequality and assuming that $\alpha<1/2$ it follows

$$S_1\le C\|f\|_{L_2(0,T)}\Big(\int_s^t (x-s)^{-2\alpha}\,dx\Big)^{1/2}=C\|f\|_{L_2(0,T)}|t-s|^{1/2-\alpha}.$$

For S_2 one infers

$$\begin{aligned}S_2&=C\int_s^t\int_s^x \frac{|f(x)-f(u)|}{(x-u)^{\alpha+1}}\,du\,dx\\ &=C\int_s^t\int_s^x \frac{|f(x)-f(u)|}{(x-u)^{\gamma+1/2}}\frac{1}{(x-u)^{\alpha-\gamma+1/2}}\,du\,dx\\ &\le C\Big(\int_s^t\int_s^x \frac{(f(x)-f(u))^2}{(x-u)^{2\gamma+1}}\,du\,dx\Big)^{1/2}\Big(\int_s^t\int_s^x \frac{1}{(x-u)^{2\alpha-2\gamma+1}}\,du\,dx\Big)^{1/2}\\ &=C\|f\|_{\widetilde{W}_2^\gamma(s,t)}(t-s)^{1/2+\gamma-\alpha}\\ &\le C\|f\|_{W_2^\gamma(0,T)}(t-s)^{1/2+\gamma-\alpha}\end{aligned}$$

and thus,

$$\begin{aligned}|h(t)-h(s)|&\le C\|f\|_{W_2^\gamma(0,T)}\|g_{t-}\|_{I_{T-}^{1-\alpha}(L_\infty(0,T))}|t-s|^{\min(1/2-\alpha,1/2+\gamma-\alpha)}\\ &=C\|f\|_{W_2^\gamma(0,T)}\|g_{t-}\|_{I_{T-}^{1-\alpha}(L_\infty(0,T))}|t-s|^{1/2-\alpha}\end{aligned}$$

for every $1-\beta<\alpha<\min(1/2,\gamma)$. Hence, the function h is Hölder continuous of all orders less than $\beta-1/2$.
(ii) The function h can be rewritten as

$$h(t)=\int_0^t D^\alpha_{0+}f_{0+}(x)D^{1-\alpha}_{t-}g_{t-}(x)\,dx+f(0+)(g(t)-g(0))$$

for all $1-\beta<\alpha<\gamma$ and so we get

$$|h(t)-h(s)|\le\Big|\int_s^t D^\alpha_{s+}f_{s+}(x)D^{1-\alpha}_{t-}g_{t-}(x)\,dx\Big|+\Big|f(s+)(g(t)-g(s))\Big|=: S_3+S_4.$$

The second summand can be estimated by

$$S_4 \leq C\|f\|_{L_\infty(0,T)}|t-s|^\beta,$$

because of the Hölder continuity of g. For the first summand we have like in (i) that

$$S_3 \leq \sup_{0\leq s<x<t\leq T} |D_{t-}^{1-\alpha} g_{t-}(x)| \int_s^t |D_{s+}^\alpha f_{s+}(x)|\, dx.$$

The last integral does not exceed

$$\begin{aligned}\int_s^t |D_{s+}^\alpha f_{s+}(x)|\, dx &\leq C\int_s^t \frac{|f(x)-f(s)|}{(x-s)^\alpha}\, dx + C\int_s^t\int_s^x \frac{|f(x)-f(u)|}{(x-u)^{\alpha+1}}\, du\, dx \\ &=: S_5 + S_6.\end{aligned}$$

Using the fact that f is Hölder continuous of order δ we obtain

$$\begin{aligned} S_5 &= C\int_s^t \frac{|f(x)-f(s)|}{(x-s)^\alpha}\, dx \\ &\leq C\int_s^t (x-s)^{\delta-\alpha}\, dx \\ &\leq C(t-s)^{1+\delta-\alpha}\end{aligned}$$

and recall that

$$S_6 \leq C\|f\|_{W_2^\gamma(0,T)}(t-s)^{1/2+\gamma-\alpha}.$$

Therefore we get

$$|h(t)-h(s)| \leq C\|f\|_{W_{2,\infty}^\gamma(0,T)}\|g_{t-}\|_{I_{T-}^{1-\alpha}(L_\infty(0,T))}|t-s|^{\min(\beta,1+\delta-\alpha,1/2+\gamma-\alpha)}.$$

for every $1-\beta<\alpha<\gamma$. Choosing α such that the minimum of $1+\delta-\alpha$ and $1/2+\gamma-\alpha$ is greater than β, this leads to the assertion.

(iii) With the same estimates for the function h as in the proof of (ii) we conclude that

$$\begin{aligned}|h(t)-h(s)| &\leq C\|f\|_{W_{2,\infty}^\gamma(0,T)}\|g_{t-}\|_{I_{T-}^{1-\alpha}(L_\infty(0,T))}|t-s|^{\min(\beta,1+\delta-\alpha,1/2+\gamma-\alpha)} \\ &= C\|f\|_{W_{2,\infty}^\gamma(0,T)}\|g_{t-}\|_{I_{T-}^{1-\alpha}(L_\infty(0,T))}|t-s|^{1/2+\gamma-\alpha}\end{aligned}$$

for all $1-\beta<\alpha<\gamma<1/2$. Hence h is Hölder continuous of order μ for all $\mu<\beta+\gamma-1/2<\beta$. □

By means of this lemma we can prove the statement about the *optimal degree of Hölder continuity.*

Theorem 5.3. *Let $g^1, ..., g^m$ be real-valued Hölder continuous functions of order $\beta > 1/2$ and consider Equation (5.1) under hypothesis* **$(\boldsymbol{H}1)$** *or* **$(\boldsymbol{H}2)$** *from Section 3.2 for the vector-valued coefficient functions σ_j, $j = 1, ..., m$. Then there exists a unique solution x to (5.1) in the sense of Theorem 3.9. Moreover, this is also the unique solution in the space of Hölder continuous functions of order β.*

Proof. The Hölder continuity of $g^1, ..., g^m$ implies $g^j_{T-} \in I^{\beta'}_{T-}(L_2(0,T))$, $j = 1, ..., m$, for any $\beta' < \beta$. Then Theorem 3.9 provides a unique solution $x \in W^{\beta-}_{2,\infty}(0,T)$. Therefore the functions $\sigma_j(x(\cdot), \cdot)$ are also elements of the space $W^{\beta-}_{2,\infty}(0,T)$ (see Proposition 2.3). Using the representation for the solution

$$x(t) = x_0 + \sum_{j=1}^{m} \int_0^t \sigma_j(x(s), s)\, dg^j(s)$$

and applying Proposition 5.2 (i) to these integrals we conclude that the function x is Hölder continuous of order $\widetilde{\beta} < \beta - 1/2$. Thus the functions $\sigma_j(x(\cdot), \cdot)$ are also Hölder continuous of order $\widetilde{\beta}$. We now apply Proposition 5.2 (ii) to the representation for the solution and obtain that x is Hölder continuous of order β.
Finally, a β-Hölder continuous function belongs to the space $W^{\beta-}_{2,\infty}(0,T)$. Therefore uniqueness follows from that in Theorem 3.9. $\square$

Lemma 5.4. *The statement of Theorem 5.3 remains valid if the condition* **$(\boldsymbol{H}1)$** *or* **$(\boldsymbol{H}2)$** *for the vector fields σ_j is replaced by (5.3) with $\delta_2 \geq \beta$ and $1 - \beta < \min(1/2, \delta_3, \frac{\delta_1}{\delta_1+1})$.*

Proof. The proof is similar as in the last theorem. The Hölder continuity of $g^1, ..., g^m$ imply that $g^j \in W^{\beta',\infty}_T(0,T)$, $j = 1, ..., m$ for any $\beta' < \beta$. By Theorem 5.1 we receive a solution x to equation (5.1) such that $x \in W^{1-\beta',\infty}_0(0,T)$ and $x \in C^{\beta'}(0,T)$. The functions $\sigma_j(x(\cdot), \cdot)$ are again elements of the space

$W_{2,\infty}^{\beta-}(0,T)$. If we apply Proposition 5.2 (i) and (ii) to the representation for the solution x we can conclude that x is Hölder continuous of order β. $\square$

In a second step we consider again the SDE (4.12) from Chapter 4, but now under the condition that w.p.1 the process Z^0 admits a generalized quadratic variation process $[Z^0]$ and has Hölder exponent $\theta \leq 1/2$ and is not Hölder continuous of some order larger than $1/2$. The real-valued stochastic processes $Z^1, ..., Z^m$ and $[Z^0]$ are Hölder continuous of order $\beta > 1/2$. For the vector fields $a_0, ..., a_m, b$ we suppose the conditions (**C1**) - (**C3**) from Chapter 4.

Theorem 5.5. *Under the above conditions, Equation (4.12) provides a pathwise unique solution X in the sense of Definition 4.20. Moreover, the solution ist Hölder continuous of the same order as the process Z^0.*

Proof. Again the Hölder continuity implies that $Z_{T-}^j \in I_{T-}^{\beta'}(L_2(0,T))$, $j = 1, ..., m$, for any $\beta' < \beta$. Therefore the proof for the existence and uniqueness of a solution is the same as in Theorem 4.21. This solution can be represented by $X(t) = h(Y(t), Z^0(t), t)$ where h is a C^1-function that solves the ODE (4.13) and Y has sample paths in $W_{2,\infty}^{\beta-}(0,T)$ and solves the auxiliary SDE (4.14). Similar as in the proof of Theorem 5.3 we can apply Proposition 5.2 (i) and (iii) to show that Y is Hölder continuous of some order $\widetilde{\beta}$ with $1/2 < \widetilde{\beta} < \beta$. This yields Hölder continuity of X with the same order as that of Z^0. $\square$

For the particular case when $m = 1$, Z^0 is a Brownian motion and the process Z^1 is a fractional Brownian motion with Hurst parameter $H > 1/2$ the result of Theorem 5.5 was obtained by Guerra and Nualart [10] under weaker regularity conditions on the coefficient functions a_j.

Chapter 6

Stochastic differential equations with jumps

In Chapter 4 we solved the stochastic differential equation (4.12) in $\mathbb{R}^n$. But all of the involved processes where continuous and also the solutions were continuous processes. For various models in physics and economics the assumption of continuity is often to restrictive. To model rare events with an immediately influence on the system jump processes are a good tool.
In the first section we recall the definition for marked point processes and in the second section we state some formulas and theorems from Chapter 4 for the discontinuous case. In Section 6.2 we extend our stochastic differential equation (4.12) by adding a component related to a marked point process. Finally we give in the last section of this chapter an example for a linear stochastic differential equation with jumps.
Througout this chapter let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

6.1 Marked point processes

Let $(T_k)_{k\geq 1}$ be an almost surely increasing sequence of random variables taking values in $\mathbb{R}_+$ with $\mathbb{P}(\lim_{k\to\infty} T_k = \infty) = 1$. We then define (see e.g. [4]) the *counting process* associated to this sequence by

$$N(t) := \sum_{k\geq 1} \mathbf{1}_{\{t\geq T_k\}}, \quad t \geq 0. \tag{6.1}$$

The process N counts the number of jumps that occur between 0 and t so that we can also write

$$N(t) = \#\{k \geq 1 : T_k \in [0,t]\}.$$

The condition $\mathbb{P}(\lim_{k\to\infty} T_k = \infty) = 1$ guarantees that N is finite for every $t \geq 0$ with probability 1, so there are only finitely many jumps. Moreover $N = (N(t))_{t\geq 0}$ is a càdlàg process which is constant between the jump times and the jump size is always 1.

If the random sequence $(T_k)_{k\geq 1}$ is defined as the sum of a sequnce of i.i.d. exponential random variables with parameter $\lambda > 0$, then N is a classical Poisson process of intensity λ.

The fact that the jump sizes of N are 1 is to restrictive for practital models. We want to allow that the jump sizes are also random. Therefore we introduce the so called *marked point processes*.

Definition 6.1. A marked point process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a double sequence $((T_k)_{k\geq 1}, (\xi_k)_{k\geq 1})$ where

(i) $(T_k)_{k\geq 1}$ is an nonanticipating almost surely increasing sequence of random variables taking values in $\mathbb{R}_+$ with $\mathbb{P}(\lim_{k\to\infty} T_k = \infty) = 1$,

(ii) $(\xi_k)_{k\geq 1}$ is a sequence of random variables taking values in $\mathbb{R}^n$ and every random variable ξ_k is $\mathcal{F}_{T_k}$ measurable.

6.2 Itô formula for processes with jumps

Let us briefly recall some definitions from Chapter 4. For two real-valued càdlàg stochastic processes X and Z and a real-valued càglàd process Y we defined the generalized stochastic forward integral by

$$\int_0^t Y\,dZ := \lim_{\substack{\varepsilon\to 0\\(ucp)}} \varepsilon \int_0^1 u^{\varepsilon-1}\int_0^t Y(s)\frac{Z_{t-}(s+u)-Z_{t-}(s)}{u}\,ds\,du$$
$$+\,Y(t)(Z(t)-Z(t-))$$

and the generalized covariation process

$$\begin{aligned}&[X,Z](t)\\&:= \lim_{\substack{\varepsilon\to 0\\(ucp)}} \varepsilon \int_0^1 u^{\varepsilon-1}\int_0^t \frac{(X_{t-}(s+u)-X_{t-}(s))(Z_{t-}(s+u)-Z_{t-}(s))}{u}\,ds\,du\\&\quad+(X(t)-X(t-))(Z(t)-Z(t-)).\end{aligned}$$

whenever the (ucp)-limit on the right-hand sides exists.

For a càdlàg process X we denote $\Delta X(t) := X(t)-X(t-)$.

As in the continuous case (see Proposition 4.13) we have that the generalized covariation is stable under C^1 transformations.

Proposition 6.2. *Let X and Z be two càdlàg processes with values in $\mathbb{R}^n$ and $\mathbb{R}^m$ admitting a generalized quadratic variation $[X],[Z]$ and the generalized covariation $[X,Z]$, and let $F:\mathbb{R}^n\to\mathbb{R}^p$ and $G:\mathbb{R}^m\to\mathbb{R}^q$ be two random C^1-functions. Then the processes $F(X)$ and $G(Z)$ admit again a generalized covariation given by*

$$\begin{aligned}[F^i(X),G^l(Z)](t) = &\sum_{j=1}^n\sum_{k=1}^m\Bigg(\int_0^t \frac{\partial F^i}{\partial x^j}(X(s-))\frac{\partial G^l}{\partial z^k}(Z(s-))\,d[X^j,Z^k](s)\\&\qquad-\sum_{0\le s\le t}\frac{\partial F^i}{\partial x^j}(X(s-))\Delta X^j(s)\frac{\partial G^l}{\partial z^k}(Z(s-))\Delta Z^k(s)\Bigg)\\&+\sum_{0\le s\le t}\Delta F^i(X(s))\Delta G^l(Z(s))\end{aligned}$$

for all $1\le i\le p$ and $1\le l\le q$.

The Itô formula for stochastic processes with jumps reads as follows (see Theorem 4.14 for the continuous case).

Theorem 6.3. *Let $Z = (Z^1, ..., Z^n)$ be a càdlàg $\mathbb{R}^n$-valued stochastic process admitting a generalized quadratic variation process $[Z]$, and let F be a random element of $C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^p)$ with continuous partial derivatives $\frac{\partial^2 F}{\partial x^j \partial x^k}, 1 \leq j,k \leq n$. Then we have for any $t > 0$*

$$\begin{aligned}
&F(Z(t),t) - F(Z(0),0) \\
=&\sum_{j=1}^{n} \int_0^t \frac{\partial F}{\partial x^j}(Z(s-),s)\, dZ^j(s) + \int_0^t \frac{\partial F}{\partial t}(Z(s-),s)\, ds \\
&+ \frac{1}{2} \sum_{j,k=1}^{n} \int_0^t \frac{\partial^2 F}{\partial x^j \partial x^k}(Z(s-),s)\, d[Z^j, Z^k](s) \\
&+ \sum_{0 \leq s \leq t} \Bigg(F(Z(s),s) - F(Z(s-),s) - \sum_{j=1}^{n} \frac{\partial F}{\partial x^j}(Z(s-),s) \Delta Z^j(s) \\
&\quad - \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 F}{\partial x^j \partial x^k}(Z(s-),s) \Delta Z^j(s) \Delta Z^k(s) \Bigg),
\end{aligned} \tag{6.2}$$

where the first n integrals are determined in the sense of generalized stochastic forward integral.

Remark 6.4. (i) Note that the sum over the jumps in the last theorem is well-defined, even if the number of jumps on the interval $[0,T]$ is infinite.

(ii) If the stochastic integral and the covariation process are defined in the sense of Russo and Vallois this formula can be found in [6].

For our stochastic differential equation in the next section we have to define as in Chapter 4 the so called general Itô formula. A solution for the SDE will be searched in the class of stochastic processes satisfying this formula.

Definition 6.5. Let $Z = (Z^1, ..., Z^m)$ be a continuous $\mathbb{R}^m$-valued stochastic process admitting a generalized quadratic variation process $[Z]$, $Y_1, ..., Y_m$ are m càglàd processes with values in $\mathbb{R}^n$ and F as in Theorem 6.3. Further let $((T_k)_{k\geq 1}, (\xi_k)_{k\geq 1})$ be a marked point process where $(\xi_k)_{k\geq 1}$ takes values in $\mathbb{R}^n$ and $N = (N(t))_{t\geq 0}$ is the counting process associated to $(T_k)_{k\geq 1}$. The $\mathbb{R}^n$-

valued process X with

$$X(t) = \sum_{j=1}^{m} \int_0^t Y_j \, dZ^j + \sum_{k=1}^{N(t)} \xi_k$$

satisfies *the general Itô formula* if for any $t > 0$

$$\begin{aligned} &F(X(t), t) - F(X(0), 0) \\ =& \sum_{i=1}^{n} \sum_{j=1}^{m} \int_0^t \frac{\partial F}{\partial x^i}(X(s), s) Y_j^i(s) \, dZ^j(s) + \int_0^t \frac{\partial F}{\partial t}(X(s), s) \, ds \\ &+ \frac{1}{2} \sum_{i,l=1}^{n} \sum_{j_1, j_2=1}^{m} \int_0^t \frac{\partial^2 F}{\partial x^i \partial x^l}(X(s), s) Y_{j_1}^i(s) Y_{j_2}^l(s) \, d[Z^{j_1}, Z^{j_2}](s) \\ &+ \sum_{k=1}^{N(t)} F(X(T_k), T_k) - F(X(T_k-), T_k). \end{aligned} \tag{6.3}$$

6.3 Result in the discontinuous case

Now we want to extend our stochastic differential equation (4.12) from Section 4.3 and include jumps. Let us consider the stochastic differential equation on $\mathbb{R}^n$:

$$\begin{aligned} X(t) =& X_0 + \sum_{j=0}^{m} \int_0^t a_j(X(s), s) \, dZ^j(s) + \int_0^t b(X(s), s) \, ds \\ &+ \sum_{k=1}^{N(t)} q(T_k, X(T_k-), \xi_k) \end{aligned} \tag{6.4}$$

for $t \in (0, T]$, where $Z^0, Z^1, \ldots, Z^m$ are real-valued continuous processes on $[0, T]$ such that Z^0 admits a generalized quadratic variation process $[Z^0]$ and the processes $[Z^0]_T, Z_T^1, ..., Z_T^m$ have sample paths in $I_{T-}^{\beta}(L_2(0, T))$ for some $1/2 < \beta < 1$. (Recall that $Z_T(t) := \mathbf{1}_{[0,T]}(t)(Z(t) - Z(T))$.) The double sequence $((T_k)_{k\geq 1}, (\xi_k)_{k\geq 1})$ is a marked point process in the sense of Definition 6.1 and the process N is a counting process associated to the sequence $(T_k)_{k\geq 1}$ which is independent of $Z^0, ..., Z^m$.

Definition 6.6. A solution of (6.4) is a stochastic process $X = (X^1, \dots, X^n)$ on $[0, T]$ which is càdlàg and has only jumps at the same time points T_k as the counting process N and is continuous between these times. Furthermore we require on X as in Definition 4.20 to admit a generalized quadratic variation process $[X]$ which satisfies the multidimensional generalized version of the Itô formula (6.3) with respect to its coordinatewise integral representation

$$\begin{aligned} X(t) = &X_0 + \sum_{j=0}^{m} \int_0^t a_j(X(s), s)\, dZ^j(s) + \int_{T_k}^t b(X(s), s)\, ds \\ &+ \sum_{k=1}^{N(t)} q(T_k, X(T_k-), \xi_k). \end{aligned}$$

For the vector fields $a_0, a_1, \dots a_m, b : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ we assume (with probability 1) as in Chapter 4:

(C1) $a_j : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ and satisfy hypothesis $(\boldsymbol{H1})$ or $(\boldsymbol{H2})$ from Section 3.2, $j = 1, ..., m$,

(C2) $a_0 \in C^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$, the partial derivatives $\frac{\partial a_0}{\partial x}(x, t)$ and $\frac{\partial a_0}{\partial t}(x, t)$ are Lipschitz in x, and $\frac{\partial a_0}{\partial x} a_0$ also satisfy hypothesis $(\boldsymbol{H1})$ or $(\boldsymbol{H2})$,

(C3) $b \in C(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$, $b(x, t)$ is Lipschitz continuous in x and Hölder continuous of order γ in t for some $\gamma > \beta$,

and for $q : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ we only assume joint measurability in all arguments.

Theorem 6.7. *Under the above conditions, there exists a unique solution to Equation (6.4).*

Proof. The counting process $N = (N(t))_{t \geq 0}$ has only finitely many jumps on $[0, T]$ and the total number of jumps will be denoted by K. The jump times are given by $T_1, ..., T_K$ and hence $0 =: T_0 < T_1 < ... < T_K < T_{K+1} := T$ is a partition of $[0, T]$.

We construct a pathwise solution to (6.4) as follows:

Theorem 4.21 implies that the SDE

$$V(t) = V_0 + \sum_{j=0}^{m} \int_{t_1}^{t_2} a_j(V(s), s)\, dZ^j(s) + \int_{t_1}^{t_2} b(V(s), s)\, ds$$

has a unique solution in the sense of Definition 4.20 for all $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. We will denote this solution by $V(t_1, t_2, V_0)$. Now we define

$${}_0X(t) := V(0, t, X_0) \quad \text{for} \quad 0 \le t \le T.$$

For ${}_1X$ we set ${}_1X(T_1) := {}_0X(T_1) + q(T_1, {}_0X(T_1), \xi_1)$ and

$${}_1X(t) := V(T_1, t, {}_1X(T_1)) \quad \text{for} \quad T_1 < t \le T.$$

In general we set for $1 \le k \le K$, if the process ${}_{k-1}X$ is already constructed, ${}_kX(T_k) := {}_{k-1}X(T_k) + q(T_k, {}_{k-1}X(T_k), \xi_k)$ and

$${}_kX(t) := V(T_k, t, {}_kX(T_k)) \quad \text{for} \quad T_k < t \le T.$$

Then the process

$$X(t) := \sum_{k=0}^{K-1} {}_kX(t) \mathbf{1}_{[T_k, T_{k+1})}(t) + {}_KX(t) \mathbf{1}_{[T_K, T]}(t)$$

is indeed a solution to (6.4).
Uniqueness follows because of the uniqueness of ${}_kX$ between the jump times and at the jump times T_k we have $\Delta X(T_k) = q(T_k, X(T_k-), \xi_k)$ for $k = 1, ..., K$. Hence any solution of Equation (6.4) coincides with the one that we constructed. □

6.4 Example

Finally we want to state an example, which may be applied for a fractional Black-Scholes model in option pricing. Let $Z^0, ..., Z^m$ be real-valued stochastic processes on $[0, T]$ with the same properties as in Section 6.3 and $Z^j(0) = 0$

for $j = 0, ..., m$. Moreover we have a marked point process $((T_k)_{k\geq 1}, (\xi_k)_{k\geq 1})$ where the sequence $(\xi_k)_{k\geq 1}$ of random variables takes values in $(-1, \infty)$ and the elements of $(T_k)_{k\geq 1}$ are independent of $Z^0, ..., Z^m$. If we consider the stochastic differential equation in $\mathbb{R}$:

$$\begin{aligned} X(t) = X_0 + \sum_{j=0}^{m} \int_0^t \sigma_j X(s)\, dZ^j(s) + \int_0^t \beta X(s)\, ds \\ + \sum_{k=1}^{N(t)} X(T_k-)\xi_k \end{aligned} \tag{6.5}$$

for some random numbers $\sigma_0, ..., \sigma_m, \beta$, then our method from the last section implies that

$$X(t) = X_0 \exp\left(\sum_{j=0}^{m} \sigma_j Z^j(t) - \frac{1}{2}\sigma_0^2 [Z^0](t) + \beta t \right) \prod_{k=1}^{N(t)} (\xi_k + 1) \tag{6.6}$$

is the unique solution to Equation (6.5).

In the continuous case this result was already presented in [42] and [40]. For $m = 0, N \equiv 0$ and a Brownian motion Z^0, Equation (6.5) is the classical Black-Scholes equation and the solution X is a geometric Brownian motion. If the marked point process is a compound Poisson process then the result for the solution of the SDE was obtained by Merton [17] (see also [4]).

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Hiermit erkläre ich,

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Bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts haben mich folgende Personen unterstützt:

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Ich habe die gleiche, in wesentlichen Teilen ähnliche bzw. eine andere Abhandlung noch bei keiner anderen Hochschule als Dissertation eingereicht.

Jena, den 20.05.2015 Erik Schneider